Loss of Coordination in Competitive Supply Chains

by

Koon Soon Teo

B.Sc. Applied Mathematics, National University of Singapore (2004)

Submitted to the School of Engineering
in partial fulfillment of the requirements for the degree of

Master of Science in Computation for Design and Optimization

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 2009

Author . . . .
School of Engineering
Aug 6, 2009

Certified by . . . .
Georgia Perakis
Professor of Operations Research
MIT Sloan School of Management
Thesis Supervisor

Accepted by . . . .
Jaime Peraire
Professor of Aeronautics and Astronautics
Director, Computation for Design and Optimization

Loss of Coordination in Competitive Supply Chains

by

Koon Soon Teo

Submitted to the School of Engineering
on Aug 6, 2009, in partial fulfillment of the
requirements for the degree of
Master of Science in Computation for Design and Optimization

Abstract

The loss of coordination in supply chains quantifies the inefficiency (i.e. the loss of total profit) due to the presence of competition in the supply chain. In this thesis, we discuss four models: one model with multiple retailers facing the multinomial logit demand, and three supply chain configurations with one supplier and multiple retailers in a i) quantity competition among retailers with substitute products, ii) price competition among retailers with substitute products, and iii) quantity competition among retailers with complement products, producing differentiated products under an affine demand function. As a special case, we also consider the symmetric setting in the four models where all retailers encounter identical demand, marginal costs, quality differences, and in the multinomial logit demand case, when there are identical variances in the consumers' utility functions.

The main contribution in this thesis lies in the precise quantification of the loss of profit due to lack of coordination, through analytical lower bounds. We provide bounds in terms of the eigenvalues of the demand sensitivity matrix, or the demand sensitivities. For the multinomial logit demand model, the lower bounds are in terms of the number of retailers and the predictability of consumer behaviour. We use simulations to provide further insights on the loss of coordination and tightness of the bounds.

We find that a supply chain with retailers operating under Bertrand competition offering substitute products is the most efficient with an average profit loss of less than 15%. We also find that competitive supply chains can be coordinated when offering substitute products. This occurs under the symmetric setting when there is a 'reasonable' number of Cournot retailers under intense competition, or when demand is 'more' inelastic in a Bertrand competition setting. As an example, in the presence of six Cournot retailers under intense competition, the profit loss is 2.04%, and when demand is perfectly inelastic in a Bertrand competition, the supply chain is perfectly coordinated with profit loss of 0%. For the multinomial logit demand case, we find that higher predictability of consumer behaviour (i.e, when consumers' choices are more deterministic) increases profits both under coordination and under competition, and a larger number of retailers decreases profits under competition, but increases profits under coordination. The net result is that efficiency 'deteriorates' when the number of competitive retailers and predictability of consumer behaviour increases.

Thesis Supervisor: Georgia Perakis
Title: Professor of Operations Research
MIT Sloan School of Management

Acknowledgments

It is a great honour to thank everyone who has made this thesis possible. First and foremost, Professor Georgia Perakis, who has been a truly extraordinary thesis advisor. I have known her to be a deeply passionate professor in her research, and a sincere mentor to me. I am very fortunate to have the opportunity to share in her expertise and experience, given her kind and patient personality. It is her commitment to her students that she spent many hours during our weekly consultations, and even more time outside our meetings to ensure that we conduct research of high quality.

I am also grateful for the strong support of the Singapore-MIT Alliance, for creating this platform and providing the funding and administrative support in our collaboration with MIT. I also wish to thank Sun Wei for her enlightening suggestions and insights into the final thesis. Not forgetting Laura Koller, who has always been looking after our needs beyond her call of duty. I must also thank my lecturers from the National University of Singapore, Prof. Koh Khee Meng, Dr. Ng Kah Loon and Dr. Tan Ban Pin for developing my love for Mathematics and inspiring me to further my education to meet the genuine needs of the society.

My most heartfelt appreciation goes out to my family, who has constantly touched me with their unreserved care and support. My father, a compassionate and humourous man, built and bonded the family through the times spent together every night. My mother, with her unconditional love and sacrifice, brought us through difficult times to give us the good life. Chun Kiat, my thoughtful brother, taught me how to be wise to the world yet principled in my values. Chun Ho, my humble brother, took care of my every possible technical and logistical needs. I might have been apart from my family for this one year, but they have always been by my side.

Finally, I dedicate this thesis to my wife, Reese, who chose to forgo everything in Singapore and followed me to the United States. We have experienced the pain of being away from family and friends, the struggles in discovering and adapting to each other, the joy of finding a place in each other's hearts. The road is long, and tough challenges continue to lie ahead. But now is the time we celebrate.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Motivation

Oligopolistic competition has been extensively researched in the economics, marketing and operations management literature. Essential questions in these researches are related to the loss of efficiency (in terms of total profit, prices and quantities) that arises when firms make individual decisions that maximize personal welfare, giving rise to outcomes that are not system optimal. This is in contrast to decisions managed by one central authority. This can lead to a substantial increase in the total profit.

The loss in efficiency due to the lack of coordination (cooperation) is well-known in the field of economics. Dubey (1986) gave the example of the Prisoner's Dilemma to show that Nash equilibria do not optimize social welfare. Papadimitriou (2001) was the first to quantify the loss of efficiency and coined the term *price of anarchy*. Price of anarchy measures how close the total profit in a competitive setting is to the total profit under cooperation. In this thesis, we consider a two-tier supply chain. This can be viewed as a Stackelberg game. We consider a single supplier who is the leader in the game, and many retailers who are the followers. The total profit in a competitive supply chain is lower than the total profit in a vertically integrated supply chain where all decisions are coordinated by a central authority. This effect of *double marginalisation* was first identified by Spengler (1950). This problem arises as a result of more than one tier in the supply chain, each tier exercising their market power, resulting in successive markups in the product's prices over marginal costs. This results in an undesirable outcome of higher prices in the market but lower seller profits, and reduces the total efficiency (measured in terms of total profit) of the competitive supply chain.

A central authority coordinating decision-making in a multi-tier supply chain, although desirable in terms of efficiency, is rarely feasible, especially when individual participants do not have incentives to comply with the central directives. As such, it is important to quantify the maximum loss of efficiency due to competition. This question arises in a variety of applications, such as transportation, auctions, facility location problems, and more recently in supply chain and revenue management settings.

1.2 Literature Review

Recent literature has begun to acknowledge the difficulty of having a central authority to coordinate decision-making. Several models, via the use of *contracts*, have been devised to design systems that give incentives to competitive firms to take actions (such as set quantities and/or prices), to reach the total profit of a coordinated setting. For a broad overview of the supply chain contracting literature, we refer the reader to Cachon (2003).

Two main types of models are considered in this literature, one of which is the newsvendor problem. In the newsvendor problem, the retailer orders from the supplier well in advance of a selling season with stochastic demand. Upon receiving the orders, the supplier begins production and delivers to the retailers at the start of the selling season. The retailers pay the supplier for every unit of order quantity, and each unit of demand above the order quantities is lost. Newsvendor problems are typically concerned with determining the optimal order quantities, or equivalently, the retailers' inventory level. Another class of models consider pricing or quantity decisions that affect the demand and the market clearing prices respectively. Therefore, firms make decisions in anticipation of the markets' response to their policies.

In a Bertrand (1883) competition, retailers compete through prices, in constrast to a Cournot (1838) competition where retailers compete through quantities. Comparison of profits between these two types of competitive settings has been discussed by Farahat and Perakis (2006).

In what follows, we first briefly review the contracting literature of newsvendor problems in supply chains. Lariviere and Porteus (2001) consider a price-only contract in a supply chain with one supplier and one retailer, and identify the relative variability, as measured by the coefficient of variation of the demand distribution, as the key driver to wholesale prices and supply chain efficiency. Cachon (2003) studies the use of various types of contracts (such as wholesale price contracts, buy-back contracts, revenue-sharing contracts, quantity-flexibility contracts and sales-rebate contracts) to coordinate supply chains that face the newsvendor problem. He considers a single supplier and a single retailer, and extends it to the case where there are multiple retailers who can choose retail prices, and is able to exert costly effort to increase demand. Cachon and Lariviere (2005) also study a revenue-sharing contract in a newsvendor problem. They consider a single supplier and a single retailer supply chain as their base model, and extended the results to one with multiple retailers. Under a revenue-sharing contract, the retailer(s) pay the supplier(s) the wholesale price plus a portion of their revenue. They demonstrate that a revenue sharing contract coordinates a supply chain with a single retailer in which the retailer chooses the optimal price and quantity, as well as a supply chain with retailers competing in terms of choosing quantities. It is well known that price-only contracts do not coordinate inventory decisions in a newsvendor problem. As such, Perakis and Roels (2007) looked into various configurations of supply chains, and quantify the loss of efficiency of a decentralized supply chain. We refer the reader to Perakis and Roels (2007) and the references within for a broad overview of the quantified loss of efficiency of various supply chain configurations.

There is also literature on the model where the market demand is dependent on the retailers' pricing or quantity policies, some involving only a single tier of retailers, others incorportating an additional layer of supplier(s). Literature that considers a single layer of competitive retailers includes the works by Farahat and Perakis (2008), who studied price competition in oligopolies without some of the commonly imposed restrictive assumptions such as, for example, homogeneous products and/or a duopoly setting. Perakis and Kluberg (2008) study the effects of a Cournot competition with multiple differentiated products on the overall society that includes firms and consumers, by measuring the total surplus and total profit under coordination and under competition under a variety of constraints. Research on the efficiency of competitive supply chains includes the work by Bernstein and Federgruen (2003), who analyze price competition among multiple retailers replenishing their inventory from a common supplier under fixed ordering costs. Subsequently, Bernstein and Federgruen (2005) extended this work to a setting under demand uncertainty, and show that coordination with multiple competing retailers under stochastic demand can be achieved by a constant wholesale-pricing scheme or price-discount sharing scheme. Goudan (2007) designed a non-coordinating contract in a single-supplier, multi-retailer supply chain where retailers make both pricing and inventory decisions. The buy back menu contract introduced improves the supply chain efficiency even in the presence of competition. Perakis and Zaretsky (2008) also studied a supply chain setting where several capacitated suppliers compete for the orders from a single retailer in a multi-period environment, and introduced option contracts to achieve a more efficient coordination in the system, while maintaining competition. More recently, Adida and DeMiguel (2009) analyzed a supply chain setting with multiple risk-averse retailers and suppliers involving multiple products, and suggested revenue-sharing contracts to improve the supply chain efficiency.

The multinomial logit demand model, a statistical model for a discrete response, arises from a probabilistic discrete choice model that describes the decision made by individuals while choosing from a discrete set of alternatives. Luce (1959) did an influential study of choice behaviour, while McFadden (1974) made a direct connection of the MNL model to consumer theory, and gave a fully consistent description of how demand is distributed. McFadden (2001) did further analysis and developed the model into what is known as the multinomial logit model today. For a brief history that led to the use of the MNL model as an important tool for microeconometric analysis of choice behaviour, we refer the reader to McFadden (2001).

The MNL model is used as the demand model in a variety of applications, including Berkovec (1985) who used it forecast automobile demand, Train, McFadden and Ben-Akiva (1987) who modeled household choices among local telephone service options, and Anderson, de Palma and Thisse (1992) who used this choice theory in product differentiation. More recently, Gaur and Honhon (2005) employed the MNL model to represent consumer demand for a planning and inventory management problem.

The widespread use of the MNL model can be attributed to its compability in explaining distribution of consumer behaviour about the mean behaviour. While traditional consumer theory using a representative agent can explain mean behaviour, it fails to adequately explain observations that deviate from the mean. As a statistical model for discrete choices, the MNL model accounts for uncertainty in the consumer utility, which can arise due to measurement errors in consumption, or consumers' error in optimization their own utility.

1.3 Thesis Outline and Main Contributions

This thesis is an extension of the thesis by Sun (2006), who quantified the price of anarchy in a single-tier Bertrand oligopoly market (consisting of multiple retailers) and proposed upper and lower bounds for the loss in efficiency due to competition.

The objective of this thesis is to consider a two-tier supply chain with one supplier and multiple retailers and understand how the presence of competition affects the total profit in the supply chain. Four models are discussed: one involving only retailers facing the MNL demand model, and three supply chain configurations involving multiple retailers in Cournot or Bertrand competition, selling differentiated *substitute* or *complement* products. Different substitute products satisfy the same needs of consumers. Therefore, consumers may prefer one to another, and make a choice among an array of substitute products. On the other hand, complement products are products that are used in conjunction with another, and have more value when consumed together. Therefore, an increase in sales of one product can usually cause a direct increase in the sales of another complement products.

To investigate the effects of competition, we employ tools from optimization to evaluate the *loss of coordination*, as a measure of the efficiency of the supply chain under competition. The loss of efficiency due to the loss of coordination is computed as the ratio of the total profit generated under competition when firms act according to selfish motivations (user optimization) and under coordination when a central authority is coordinating all decision-making (system optimization). We also use matrix theory to analyze the equilibrium prices, equilibrium production quantities, the retailers' profits and the supplier's profits under user optimization and system optimization respectively, and quantify the loss of efficiency. We then present easily computable lower bounds for the loss of coordination without some of the commonly imposed restrictive assumptions in the literature such as, for example, homogeneous products and/or a duopoly setting. We prove analytically the tightness of these bounds, and use simulations to give further insights on the efficiency of the supply chain, and show that the actual loss of coordination is, in fact, 'very close' to our derived bounds for an overwhelming majority of randomly generated data instances.

The bounds we present for the competitive supply chain settings are dependent only on two key drivers - the number of retailers and the price (or quantity) sensitivity for Bertrand competition (or Cournot competition, respectively). *Price sensitivity* quantifies the change in market demand due to changes in the retail prices, while *quantity sensitivity* quantifies the change in market clearing prices due to changes in market supply. As a special case, we also consider the symmetric setting under uniform demand without quality differences among products from different sellers. In a symmetric setting, all retailers encounter identical price (or quantity) sensitivities and the same demand function for all their products.

We find that a supply chain with retailers operating under Bertrand competition offering substitute products is the most efficient with an average profit loss of less than 15%. On the other hand, a supply chain with retailers operating under Cournot competition offering substitute products has an average profit loss of less than 30%.

We also find that competitive supply chains can be coordinated when offering substitute products. This occurs under the symmetric setting when there is a 'reasonable' number of Cournot retailers under intense competition, or when demand is 'more' inelastic (i.e., demand is not 'significantly' affected by prices) in a Bertrand competition setting. As an example, in the presence of six Cournot retailers under intense competition, the profit loss is 2.04%, and when demand is perfectly inelastic in a Bertrand competition, the supply chain is perfectly coordinated with profit loss of 0%. However, we must highlight that, despite high efficiencies under such circumstances, the profit of the monopolistic supplier is very large compared to the total profit of the retailers. Thus, the supplier dominates the total profit.

In the last model we study in this thesis, we consider a single tier of price-competing retailers facing the multinomial logit demand function which is derived from a probabilistic consumer utility demand function. We evaluate the loss of coordination and propose lower bounds to quantify the efficiency of these retailers under competition. As a special case, we consider a symmetric setting where all retailers encounter identical marginal costs, quality differences and variances in the probabilistic component of the consumer utility function. Simulations are conducted to evaluate the tightness of these bounds and to discuss further insights on the loss of coordination under the multinomial logit demand. We identified two key drivers to profits and efficiency - the number of retailers and the predictability of consumer behaviour. Consumer behaviour is said to be more predictable if the consumer utility function is more deterministic. We find that higher predictability of consumer behaviour (i.e, when consumers' choices are more deterministic) increases profits both under coordination and under competition, and a larger number of retailers decreases profits under competition, but increases profits under coordination. The net result is that efficiency 'deteriorates' when the number of competitive retailers and predictability of consumer behaviour increases.

The structure of the remainder of this thesis is as follows. Chapter 2 studies a supply chain with one supplier and many retailers, the latter operating under Cournot competition offering substitute products. Chapter 3 deals with a similar supply chain but where retailers operate under Bertrand competition offering substitute products. Chapter 4 studies a similar supply chain with retailers in a Cournot competition offering complement products. Chapter 5 considers the setting with one tier supply chain with many retailers facing the multinomial logit demand model. Analysis of the loss of coordination, simulation results in the asymmetric and the symmetric settings for the different models can be found in their respective chapters. Chapter 6 discusses conclusions and open questions for future research.

Chapter 2

Cournot Competition with Substitute Products

2.1 Overview and Main Contributions

In this chapter, we analyze the loss of profit due to lack of coordination (we refer to this as loss of coordination) in a single-supplier, multi-retailer supply chain setting. The supply chain we consider is a Stackelberg game where the supplier is the leader and the retailers are the followers. The retailers compete in an oligopoly market through deciding quantities (Cournot competition) of substitute products. Our model considers an affine demand price relation. This arises naturally from a quasilinear consumer utility function. As a special case, we also consider a uniform demand function, when all retailers encounter identical demand (i.e., have the same quantity sensitivities for all products). The demand function represents the consumers in an aggregate format and depends only on the quantities set by the retailers.

We evaluate the loss of coordination to measure the efficiency of the supply chain under competition, computed as the ratio of the total profit (that is, the total supplier's and retailers' profit) generated under competition (user optimization) and under coordination (system optimization). We then propose lower bounds for this loss of coordination to quantify the efficiency of the supply chain under competition. The lower bounds are in terms of the eigenvalues of the demand sensitivity matrix, or the demand sensitivities. In addition, we conduct simulations which further indicate that the average loss due to competition of the supply chain is no more than 30%, implying that the competitive (uncoordinated) supply chain is in fact fairly efficient. Moreover, theoretical and simulation results both indicate that under uniform demand, the supply chain can be 'almost' coordinated when there is a 'reasonable' number of (e.g., six or more) retailers under intense competition in the market. For example, the loss of efficiency due to 'very' intense competition is 1.23%, 2.04% and 4% in the presence of eight, six and four retailers respectively.

The structure of the remainder of this chapter is as follows. Section 2.2 provides the groundwork for this chapter. Subsection 2.2.1 gives the notations and assumptions imposed in our analysis. We discuss the rationale and validity of these assumptions. In Subsection 2.2.2, we list several important definitions including the central concept of Nash equilibrium. In Subsection 2.2.3, we describe the model and review the central concepts of Nash equilibrium, user optimum, system optimum and the loss of coordination. Section 2.3 presents the equilibrium wholesale prices, market clearing prices, order quantities and total profits under user optimization, when individual market participants maximize their own profits. In Section 2.4, we derive the optimal market clearing prices, order quantities and total profits achieved under system optimization, when a central authority is coordinating decisions. Section 2.5 presents the most important findings in this chapter - the loss of coordination in terms of the quantity sensitivity matrix in Subsection 2.5.1, and presents lower bounds for this loss of coordination. We present three lower bounds, one in Subsection 2.5.2 which is in terms of the minimum eigenvalue of the quantity sensitivity matrix, and two lower bounds in Subsection 2.5.3 in terms of the quantity sensitivity ratio, which are easier to compute. Simulations are performed in Section 2.6 to evaluate and compare the tightness of these bounds. Numerical results from these simulations also indicate that the average loss due to competition in the supply chain is no more than 30%. Finally, in Section 2.7, we analyze the loss of coordination under the uniform demand model, i.e., when all retailers encounter identical quantity sensitivities and experience the same demand function for all their products. Results under the uniform demand model indicate that greater intensity of competition among retailers leads to higher efficiency under competition, with the possibility of attaining close to no loss in efficiency when there is a large number of retailers competiting in the market.

2.2 Preliminaries

We consider a two tier single-supplier, multi-retailer supply chain producing differentiated substitute products under an affine price demand relation competiting in a Cournot (quantity) oligopoly market, where retailers compete by deciding the quantities to produce and sell to the market at market clearing prices. We will first list the associated notations and assumptions in Section 2.2.1, and give more specific details of the model in Section 2.2.3.

2.2.1 Assumptions and Notations

In this supply chain with a single supplier and n retailers, we denote the order quantity of retailer i ($i = 1, 2, ..., n$) by q_i and let vector $\mathbf{q} = (q_1, ..., q_n)^T$. Similarly, let vectors $\mathbf{d}$, $\mathbf{c}$, $\mathbf{w}$ and $\mathbf{p}$ be the respective vectors for the market clearing prices under zero production, the costs per unit order incurred by the supplier, the wholesale prices charged by the supplier and the market clearing prices. Let Z_{R_i} be the profit of retailer i, and Z_S be the supplier's profit.

Let the equilibrium wholesale prices, market clearing prices, production quantities and total profits under competition (user optimization) be denoted by $\mathbf{w_{UO}}$, $\mathbf{p_{UO}}$, $\mathbf{q_{UO}}$ and Z_{UO} respectively. Let the optimal market clearing prices, production quantities and total profits under coordination (system optimization) be denoted by $\mathbf{p_{SO}}$, $\mathbf{q_{SO}}$ and Z_{SO} respectively.

Our analysis is restricted to models that satisfy the following assumptions:

Assumption 2.2.1 *The price demand relationship is affine and deterministic.*

Affine demand functions are common in the pricing literature. Such a model arises naturally from a quasilinear utility function of a representative consumer. This model has been used by many researchers such as Carr et al. (1999), Berstein and Federgruen (2003), Allon and Federgruen (2006, 2007). In this thesis, we remove the effects of stochasticity of demand in order to isolate the effects of competition.

Given the production quantity vector $\mathbf{q}$, the market clearing price vector $\mathbf{p}$ is obtained from the price-demand function as follows:

$$\begin{aligned}\mathbf{q} &= \overline{\mathbf{d}} - \overline{\mathbf{B}}\mathbf{p},\\ \mathbf{p} &= \overline{\mathbf{B}}^{-1}\overline{\mathbf{d}} - \overline{\mathbf{B}}^{-1}\mathbf{q} = \mathbf{d} - \mathbf{B}\mathbf{q},\end{aligned} \tag{2.1}$$

where $\mathbf{d} = \overline{\mathbf{B}}^{-1}\overline{\mathbf{d}}$ and $\mathbf{B} = \overline{\mathbf{B}}^{-1}$.

Assumption 2.2.2 *The inverse of the quantity sensitivity matrix, $\overline{\mathbf{B}}$ (which is $\mathbf{B}^{-1}$), is a symmetric matrix.*

As a result, the quantity sensitivity matrix $\mathbf{B}$ is also symmetric. This assumption implies that the cross-effects of the retailers' production quantities on each other are symmetric. This model arises naturally when a representative consumer maximizes a quasilinear utility function.

Assumption 2.2.3 *Matrix $\overline{\mathbf{B}}$ has positive diagonals and non-positive off-diagonals.*

This is a natural consequence of a market with substitute products. Increasing a retailer's selling price has a negative effect on its own market demand, but a non-negative effect on other retailers' market demand.

Assumption 2.2.4 *$\overline{\mathbf{B}}$ is a diagonally dominant matrix.*

This implies that a retailer's policy has a higher effect on its market demand than the total effect of the prices of all other retailers.

Assumption 2.2.5 *The following relation holds:*

$$(\mathbf{B} + \mathbf{\Gamma})^{-1}(\mathbf{d} - \mathbf{c}) \geq 0.$$

Assumptions 2.2.3 and 2.2.4 imply that $\mathbf{B}$ is an inverse M-matrix (see definition in Subsection 2.2.2). As a result, $\mathbf{d} \geq \mathbf{c}$, and

$$\mathbf{B}^{-1}(\mathbf{d} - \mathbf{c}) \geq 0.$$

As we will see later in this chapter, this implies that all retailers' demand, hence the order quantities, are non-negative. This requires the vector of prices at zero demand, $\mathbf{d}$, and the vector of costs incurred by the supplier, $\mathbf{c}$, to be such that it is profitable to supply and sell a non-negative amount of the product. This assumption is valid because products which do not satisfy this requirement are not profitable to produce and naturally do not exist in the market under user optimization and system optimization. See also Adida and DeMiguel (2009) who also impose and discuss this assumption.

Note that when demand is uniform (see Section 2.7.1 for definition), this assumption is equivalent to $d \geq c$. This implies that prices at zero quantities are greater than or equal to marginal costs.

Assumption 2.2.6 *The marginal costs of production are non-negative. That is,* $\mathbf{c} \geq 0$.

From Assumption 2.2.5, this also implies that $\mathbf{d} \geq 0$.

Let $\overline{\mathbf{B}}$ be the following matrix:

$$\overline{\mathbf{B}} = \begin{bmatrix} \alpha_1 & -\beta_{1,2} & \cdots & \cdots & -\beta_{1,n} \\ -\beta_{2,1} & \alpha_2 & \cdots & \cdots & -\beta_{2,n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -\beta_{n-1,1} & \vdots & \ddots & \alpha_{n-1} & -\beta_{n-1,n} \\ -\beta_{n,1} & \cdots & \cdots & -\beta_{n,n-1} & \alpha_n \end{bmatrix},$$

and let $\mathbf{\Gamma}$ be a diagonal matrix consisting only of the diagonals of matrix $\overline{\mathbf{B}}^{-1}$.

Remark Assumption 2.2.3 requires $\alpha_i > 0$ and $\beta_{i,j} \geq 0$ for all i, j. Assumption 2.2.4

requires $|\alpha_i| \geq \sum_{i \neq j} |\beta_{i,j}|$ for all $i,\ j$. These assumptions imply that $\overline{\mathbf{B}}$ and $\mathbf{B}$ are an M-matrix and inverse M-matrix respectively, defined in the following section.

2.2.2 Definitions

We list some important definitions and its associated results that will be used in this chapter.

Definition *(Nash Equilibrium)* The decisions for each player are *Nash equilibrium policies* if no single player can increase his payoff by unilaterally changing his policy.

Definition *(Uniform demand)* In a market under uniform demand, all firms encounter identical price sensitivities (under Bertrand competition), quantity sensitivities (under Cournot competition) and experience the same demand function.

Definition *(M-matrix)* A matrix $\mathbf{A}$ is called an M-matrix if $\mathbf{A} \in \mathbf{Z_n}$ and $\mathbf{A}$ is positive stable (i.e., every eigenvalue has positive real part), where $\mathbf{Z_n} = \{\mathbf{A} = [a_{ij}] \in M_n(\Re) : a_{ij} \leq 0 \text{ if } i \neq j, \quad i, j = 1, ..., n\}$.

Definition *(Inverse M-matrix)* A matrix $\overline{\mathbf{A}}$ is called an inverse M-matrix if $\overline{\mathbf{A}} = \mathbf{A}^{-1}$, where $\mathbf{A}$ is an M-matrix.

Remark The following results will be useful in this chapter. We refer the reader to Horn and Johnson (1985) for the proof of these properties.

1. Let $\mathbf{A} \in \mathbf{Z_n}$. The following statements are equivalent.

    (a) $\mathbf{A}$ is an M-matrix.

    (b) Every real eigenvalue of $\mathbf{A}$ is positive.

    (c) $\mathbf{A}$ is nonsingular and $\mathbf{A}^{-1} \geq 0$.

    (d) The diagonal entries of $\mathbf{A}$ are positive and there exists positive diagonal matrices $\mathbf{D}$, $\mathbf{E}$ such that $\mathbf{DAE}$ is both strictly row diagonally dominant and strictly column diagonally dominant.

2. Let $\mathbf{A}, \mathbf{B} \in \mathbf{Z_n}$, where $\mathbf{A}$ is an M-matrix and $\mathbf{B} \geq \mathbf{A}$. Then

   (a) $\mathbf{B}$ is an M-matrix.

   (b) $\mathbf{A}^{-1} \geq \mathbf{B}^{-1} \geq 0$

Definition *(Quantity sensitivity ratio)* The quantity sensitivity ratio, $r_i(\mathbf{B})$, for retailer i is obtained from the quantity sensitivity matrix $\mathbf{B}$. It is defined as

$$r_i(\mathbf{B}) = \sum_{i \neq j} \frac{|\beta_{i,j}|}{|\alpha_i|}.$$

The definition of the quantity sensitivity ratio can also be extended to one obtained from the normalized quantity sensitivity matrix $\mathbf{G}$, where $\mathbf{G} = \mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{B}\mathbf{\Gamma}^{-\frac{1}{2}}$.

Definition *(Similar matrices)* Let $\mathbf{A}, \mathbf{B} \in \mathbf{Z_n}$. $\mathbf{A}$ and $\mathbf{B}$ are similar matrices if there exists an invertible $n \times n$ matrix $\mathbf{P}$ such that $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

Remark Similar matrices have the same set of eigenvalues.

2.2.3 Model Description

We consider a two tier single-supplier, multi-retailer supply chain producing differentiated substitute products under an affine demand function.

The sequence of events is as follows. The supplier is a Stackelberg leader who first proposes a wholesale price to each of the retailers. After receiving the wholesale price, each retailer makes a decision on their own order quantities, and specifies to the supplier his/her respective order quantity. Upon receiving the order quantities, the supplier begins production and delivers items to each retailer at costs incurred by the supplier. The representative consumer will pay for all products available and therefore all quantities ordered by the retailers will be sold to the market.

In a Cournot (quantity) oligopoly market, the retailers compete by deciding the quantity to produce and sell to the market at market clearing prices, which are determined as functions of the quantities sold through the inverse demand function (see Equation (2.1)).

Under user optimization, the supplier maximizes her profit by deciding the wholesale prices as a best response to the anticipated equilibrium order quantities by the retailers. The retailers decide on the quantities to sell to the market in response to the supplier's pricing policy. The supplier and each retailer is assumed to be rational and selfish, optimizing profits only for themselves. Nash Equilibrium is reached when no single retailer can increase its profit by unilaterally changing its production quantity.

For each retailer i, given the supplier's equilibrium wholesale price, w_i obtained from the vector $\mathbf{w_{UO}}$, and competitors' equilibrium quantities given by the vector $\mathbf{q_{UO,-i}}$, the retailer's quantity policy is obtained by solving the optimization problem $\mathbf{UO_{R_i}}$ described as follows:

$$\begin{aligned} \mathbf{UO_{R_i}}: \quad & \max_{q_i} \quad q_i.(p_i(q_i, \mathbf{q_{UO,-i}}) - w_i), \\ & \text{s.t.} \quad q_i \geq 0, p_i \geq w_i \end{aligned} \tag{2.2}$$

The equilibrium wholesale price, w_i, for retailer i in the above problem is the solution to the supplier's optimization problem. The supplier maximizes profit by deciding the wholesale price vector, $\mathbf{w_{UO}}$, given the retailers' equilibrium quantities obtained from vector $\mathbf{q_{UO}}$. This optimization problem, $\mathbf{UO_S}$, is described as follows:

$$\begin{aligned} \mathbf{UO_S}: \quad & \max_{\mathbf{w}} \quad \sum_{i=1}^{n} (w_i - c_i).q_i(w_i, \mathbf{w_{UO,-i}}), \\ & \text{s.t.} \quad q_i \geq 0, w_i \geq c_i \quad \text{for all} \quad i = 1, ..., n. \end{aligned} \tag{2.3}$$

Let $Z_{R_i}^{UO}$ denote the profit of retailer i obtained from Optimization Problem (2.2), and Z_S^{UO} be the profit of the supplier obtained from Optimization Problem (2.3). The total profit under user optimization, Z_{UO}, is the sum of the profits of all the retailers and the supplier given by

$$Z_{UO} = Z_S^{UO} + \sum_{i=1}^{n} Z_{R_i}^{UO}.$$

Under system optimization, a central authority is coordinating all decisions, optimizing the total profit of the supplier and all retailers. The central authority makes decisions on all production quantities and forces the supplier and all retailers to comply. Coordination is attained by solving the following optimization problem, which determines the production quantities that maximize the total supply chain profit of the supplier and all retailers.

$$\begin{aligned} \textbf{SO:} \quad \max_{\mathbf{q_{SO}}} \quad & Z_S + \sum_{i=1}^{n} Z_{R_i}, \\ \text{s.t.} \quad & \mathbf{q_{SO}} \geq 0, \mathbf{p_{SO}} \geq \mathbf{c} \end{aligned} \tag{2.4}$$

Let Z_{SO} denote the optimal total profit obtained by solving the above optimization problem.

The loss of coordination, LOC, measures the loss of the total supply chain profit under competition, computed as the ratio of the total profit generated under user optimization and under system optimization. That is,

$$LOC = \frac{Z_{UO}}{Z_{SO}}. \tag{2.5}$$

2.3 User Optimization

In this section, we will derive the equilibrium production quantities, wholesale prices, market clearing prices and the total profits under user optimization.

We first relax the non-negativity constraints and solve the first order optimality conditions for the optimization problems. We then show that, under the assumptions we impose in Section 2.2.1, the solutions obtained satisfy the constraints and are thus feasible, and hence optimal, solutions to Optimization Problems (2.2) and (2.3).

Proposition 2.3.1 *Under Assumptions 2.2.1 and 2.2.2, the equilibrium total profit in user optimization is*

$$Z_{UO} = \frac{1}{4}(\mathbf{d}-\mathbf{c})^T[(\mathbf{B}+\boldsymbol{\Gamma})^{-1} + (\mathbf{B}+\boldsymbol{\Gamma})^{-1}\boldsymbol{\Gamma}(\mathbf{B}+\boldsymbol{\Gamma})^{-1}](\mathbf{d}-\mathbf{c}). \tag{2.6}$$

There exist unique equilibrium wholesale prices, $\mathbf{w_{UO}}$*, production quantities,* $\mathbf{q_{UO}}$*,* *and market clearing prices, $\mathbf{p_{UO}}$, given respectively by*

$$\mathbf{w_{UO}} = \frac{1}{2}(\mathbf{d} + \mathbf{c}).$$

$$\mathbf{q_{UO}} = \frac{1}{2}(\mathbf{B} + \mathbf{\Gamma})^{-1}(\mathbf{d} - \mathbf{c}), \tag{2.7}$$

$$\mathbf{p_{UO}} = \mathbf{d} - \frac{1}{2}\mathbf{B}(\mathbf{B} + \mathbf{\Gamma})^{-1}(\mathbf{d} - \mathbf{c}). \tag{2.8}$$

Proof Given the wholesale price, w_i, imposed by the supplier, each retailer i makes a decision on the quantity to order from the supplier, so as to maximize their own profit. Under an affine demand function, the market clearing price for retailer i is

$$p_i(q_i, \mathbf{q_{UO,-i}}) = d_i - \alpha_i q_i + \sum_{j \neq i} \beta_{i,j} q_j^{UO}.$$

With the above market clearing price, the profit for retailer i is

$$Z_{R_i} = q_i(d_i - \alpha_i q_i + \sum_{j \neq i} \beta_{i,j} q_j - w_i).$$

We relax the non-negativity constraints in Optimization Problem (2.2) to determine the best response quantity policy for retailer i, which is achieved when

$$\frac{\partial Z_{R_i}}{\partial q_i} = d_i - 2\alpha_i q_i + \sum_{j \neq i} \beta_{i,j} q_j - w_i = 0,$$

$$\nabla \mathbf{Z_R}(\mathbf{q}) = \mathbf{d} - \mathbf{Bq} - \mathbf{\Gamma q} - \mathbf{w} = 0.$$

Therefore, the retailers' order quantity is

$$\mathbf{q} = (\mathbf{B} + \mathbf{\Gamma})^{-1}(\mathbf{d} - \mathbf{w}),$$

given the supplier's wholesale price vector $\mathbf{w}$. In particular, at Nash equilibrium, given the optimal wholesale price vector $\mathbf{w_{UO}}$, the equilibrium quantity is

$$\mathbf{q_{UO}} = (\mathbf{B} + \mathbf{\Gamma})^{-1}(\mathbf{d} - \mathbf{w_{UO}}). \tag{2.9}$$

Given the equilibrium order quantity from Equation (2.9), the supplier makes a decision on the wholesale price to maximize his profit. The supplier's profit, $\mathbf{Z_S}$, is given by:

$$Z_S = (\mathbf{w} - \mathbf{c})^T \mathbf{q_{UO}} = (\mathbf{w} - \mathbf{c})^T (\mathbf{B} + \mathbf{\Gamma})^{-1} (\mathbf{d} - \mathbf{w}).$$

We relax the non-negativity constraints in Optimization Problem (2.3) given the anticipated production quantities, to determine the supplier's optimal pricing policy. Optimality is achieved when

$$\nabla \mathbf{Z_S}(\mathbf{w}) = (\mathbf{B} + \mathbf{\Gamma})^{-T} \mathbf{c} + (\mathbf{B} + \mathbf{\Gamma})^{-1} \mathbf{d} - [(\mathbf{B} + \mathbf{\Gamma})^{-T} + (\mathbf{B} + \mathbf{\Gamma})^{-1}] \mathbf{w} = 0.$$

Therefore, the supplier's optimal wholesale price is

$$\mathbf{w_{UO}} = [(\mathbf{B} + \mathbf{\Gamma})^{-T} + (\mathbf{B} + \mathbf{\Gamma})^{-1}]^{-1} [(\mathbf{B} + \mathbf{\Gamma})^{-T} \mathbf{c} + (\mathbf{B} + \mathbf{\Gamma})^{-1} \mathbf{d}]. \tag{2.10}$$

Under Assumption 2.2.2, which requires $\mathbf{B}$ to be a symmetric matrix, it follows that

$$(\mathbf{B} + \mathbf{\Gamma})^{-T} = (\mathbf{B} + \mathbf{\Gamma})^{-1}.$$

The supplier's optimal wholesale price therefore reduces to

$$\begin{aligned}
\mathbf{w_{UO}} &= [2(\mathbf{B} + \mathbf{\Gamma})^{-1}]^{-1} [(\mathbf{B} + \mathbf{\Gamma})^{-1} (\mathbf{d} + \mathbf{c})], \\
&= \frac{1}{2} (\mathbf{B} + \mathbf{\Gamma}) (\mathbf{B} + \mathbf{\Gamma})^{-1} (\mathbf{d} + \mathbf{c}), \\
&= \frac{1}{2} (\mathbf{d} + \mathbf{c}), && (2.11) \\
&\geq \mathbf{c} \quad \text{By Assumption 2.2.5.} && (2.12)
\end{aligned}$$

From the retailers' equilibrium quantity policies and the supplier's equilibrium pricing policy in Equation (2.9) and Equation (2.10), we can express the equilibrium quantities, prices and profits in terms of constant vectors, $\mathbf{d}$ and $\mathbf{c}$, and constant matrices, $\mathbf{B}$ and $\mathbf{\Gamma}$.

Substituting Equation (2.10) into Equation (2.9), we obtain the equilibrium quantities under user optimization, given by the vector

$$\mathbf{q_{UO}} = (\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{d} - [(\mathbf{B}+\mathbf{\Gamma})^{-T} + (\mathbf{B}+\mathbf{\Gamma})^{-1}]^{-1}[(\mathbf{B}+\mathbf{\Gamma})^{-T}\mathbf{c} + (\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{d}]).$$

When $\mathbf{B}$ is a symmetric matrix, we substitute Equation (2.11) into Equation (2.9), and obtain

$$\begin{aligned}\mathbf{q_{UO}} &= (\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{d} - \frac{1}{2}\mathbf{d} - \frac{1}{2}\mathbf{c})],\\ &= \frac{1}{2}(\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{d}-\mathbf{c}).\end{aligned}$$

By Assumption 2.2.5, $\mathbf{q_{UO}} \geq 0$, and is therefore a feasible solution. The equilibrium market clearing price, $\mathbf{p_{UO}}$, can be obtained from Equation (2.7) and the price demand relationship in Assumption 2.2.1, as shown below:

$$\begin{aligned}\mathbf{p_{UO}} &= \mathbf{d} - \mathbf{B}\mathbf{q_{UO}},\\ &= \mathbf{d} - \mathbf{B}(\frac{1}{2}(\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{d}-\mathbf{c})),\\ &= \mathbf{d} - \frac{1}{2}\mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{d}-\mathbf{c}),\\ &\geq \mathbf{w_{UO}}.\end{aligned}$$

The optimal total profit generated in the market, Z_{UO}, is the sum of the profits of the supplier and all retailers given by

$$\begin{aligned}
Z_{UO} &= (\mathbf{p_{UO}} - \mathbf{w_{UO}})^T \mathbf{q_{UO}} + (\mathbf{w_{UO}} - \mathbf{c})^T \mathbf{q_{UO}}, \\
&= (\mathbf{p_{UO}} - \mathbf{c})^T \mathbf{q_{UO}}, \\
&= [\mathbf{d} - \frac{1}{2}\mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{d}-\mathbf{c}) - \mathbf{c}]^T [\frac{1}{2}(\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{d}-\mathbf{c})], \\
&= \frac{1}{4}[2(\mathbf{d}-\mathbf{c}) - \mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{d}-\mathbf{c}) - \mathbf{c}]^T [(\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{d}-\mathbf{c})], \\
&= \frac{1}{4}(\mathbf{d}-\mathbf{c})^T [2\mathbf{I} - (\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{B}](\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{d}-\mathbf{c}), \\
&= \frac{1}{4}(\mathbf{d}-\mathbf{c})^T [2\mathbf{I} - (\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{B}+\mathbf{\Gamma}) + (\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma}](\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{d}-\mathbf{c}), \\
&= \frac{1}{4}(\mathbf{d}-\mathbf{c})^T [2\mathbf{I} - \mathbf{I} + (\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma}](\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{d}-\mathbf{c}), \\
&= \frac{1}{4}(\mathbf{d}-\mathbf{c})^T [(\mathbf{B}+\mathbf{\Gamma})^{-1} + (\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma}(\mathbf{B}+\mathbf{\Gamma})^{-1}](\mathbf{d}-\mathbf{c}).
\end{aligned}$$

∎

Remark We have shown that the solutions $\mathbf{q_{UO}}$ and $\mathbf{w_{UO}}$ satisfy the first order optimality conditions when there are no non-negativity constraints. Nevertheless, the non-negativity constraints due to Assumption 2.2.5. They are thus feasible solutions to the retailers' and supplier's optimization problems respectively.

2.4 System Optimization

We will derive the optimal prices, quantities and profits under system optimization.

We will relax the constraints and solve the first order optimality condition for the objective function. We will then show that the solutions obtained satisfy the constraints and are thus feasible solutions to the optimization problem.

Proposition 2.4.1 *Under Assumptions 2.2.1 and 2.2.2, the optimal total profit generated by the system optimization is*

$$Z_{SO} = \frac{1}{4}(\mathbf{d}-\mathbf{c})^T \mathbf{B}^{-1}(\mathbf{d}-\mathbf{c}). \tag{2.13}$$

There exist unique optimal production quantities, $\mathbf{q_{SO}}$*, and market clearing prices,* $\mathbf{p_{SO}}$*, given respectively by*

$$\begin{aligned} \mathbf{q_{SO}} &= \frac{1}{2}\mathbf{B}^{-1}(\mathbf{d} - \mathbf{c}), \\ \mathbf{p_{SO}} &= \frac{1}{2}(\mathbf{d} + \mathbf{c}). \end{aligned} \tag{2.14}$$

Notice these are independent from the wholesale price.

Proof The total profit of the system is the sum of retailers' profits, Z_{R_i} for retailer i, and the supplier's profit, Z_S. It is given by

$$\begin{aligned} Z &= Z_S + \sum_{i=1}^{n} Z_{R_i}, \\ &= (\mathbf{w} - \mathbf{c})^T \mathbf{q} + (\mathbf{p}(\mathbf{q}) - \mathbf{w})^T \mathbf{q}, \\ &= (\mathbf{p}(\mathbf{q}) - \mathbf{c})^T \mathbf{q}. \end{aligned}$$

The resulting optimization problem is

$$\begin{aligned} \textbf{SO:} \quad \max_{\mathbf{q_{SO}}} \quad & (\mathbf{p}(\mathbf{q_{SO}}) - \mathbf{c})^T \mathbf{q_{SO}}, \\ \text{s.t.} \quad & \mathbf{q_{SO}} \geq 0, \quad \mathbf{p}(\mathbf{q_{SO}}) \geq 0. \end{aligned} \tag{2.15}$$

Under an affine demand function, Z_{SO} is given by

$$Z_{SO} = (\mathbf{d} - \mathbf{B}\mathbf{q_{SO}} - \mathbf{c})^T \mathbf{q_{SO}}. \tag{2.16}$$

Optimality under no non-negativity constraints is achieved when

$$\nabla \mathbf{Z}(\mathbf{q_{SO}}) = \mathbf{d} - (\mathbf{B} + \mathbf{B}^T)\mathbf{q_{SO}} - \mathbf{c} = 0, \tag{2.17}$$

$$\mathbf{q_{SO}} = (\mathbf{B} + \mathbf{B}^T)^{-1}(\mathbf{d} - \mathbf{c}). \tag{2.18}$$

Under Assumption 2.2.2, the optimal production quantities are given by the vector

$$\mathbf{q_{SO}} = \frac{1}{2}\mathbf{B}^{-1}(\mathbf{d} - \mathbf{c}).$$

By Assumption 2.2.5, $\mathbf{q_{SO}} \geq 0$, and is therefore a feasible solution to the constrained problem. The equilibrium market clearing price vector is

$$\begin{aligned}
\mathbf{p_{SO}} &= \mathbf{d} - \mathbf{B}[\frac{1}{2}\mathbf{B}^{-1}(\mathbf{d} - \mathbf{c})], \\
&= \frac{1}{2}(\mathbf{d} + \mathbf{c}), \\
&\geq \mathbf{c}.
\end{aligned}$$

Substituting Equation (2.18) into Equation (2.16), we obtain the optimal total profit generated by the system optimization, given by

$$\begin{aligned}
Z_{SO} &= [\mathbf{d} - \mathbf{B}(\frac{1}{2}\mathbf{B}^{-1}(\mathbf{d} - \mathbf{c})) - \mathbf{c}]^T[\frac{1}{2}\mathbf{B}^{-1}(\mathbf{d} - \mathbf{c})], \\
&= \frac{1}{4}(\mathbf{d} - \mathbf{c})^T\mathbf{B}^{-1}(\mathbf{d} - \mathbf{c})
\end{aligned}$$

∎

Remark 1. The wholesale price vector $\mathbf{w}$, is a internal transaction between the retailers and the supplier. Under system optimization, this transaction is executed within the system and therefore has no influence on system profits.

2. We have shown that the solution $\mathbf{q_{SO}}$ satisfies the first order optimality conditions, and the non-negativity constraint due to Assumption 2.2.5. Therefore, $\mathbf{q_{SO}}$ and $\mathbf{p_{SO}}$ are feasible solutions to the system optimization problem.

2.5 Loss of Coordination under a General Affine Demand Model

We first derive the exact loss of coordination in terms of the quantity sensitivity matrix. Subsequently we give three lower bounds for the loss of coordination: One in terms of the minimum eigenvalue of the normalized quantity sensitivity matrix, and two in terms of the quantity sensitivity ratio.

2.5.1 Loss of Coordination in terms of the Quantity Sensitivity Matrix

In the following lemma, we will express the loss of coordination in terms of $\mathbf{G}$ and $\mathbf{w}$, where

$$\mathbf{G} = \mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{B}\mathbf{\Gamma}^{-\frac{1}{2}},$$

$$\mathbf{w} = \mathbf{\Gamma}^{\frac{1}{2}}(\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{d}-\mathbf{c}).$$

Lemma 2.5.1 *Under Assumptions 2.2.1 and 2.2.2, the loss of coordination in a Cournot competition with substitute products is given by:*

$$LOC = \frac{\mathbf{w}^T(\mathbf{G}+2\mathbf{I})\mathbf{w}}{\mathbf{w}^T(\mathbf{G}+\mathbf{G}^{-1}+2\mathbf{I})\mathbf{w}}, \tag{2.19}$$

where $\mathbf{G} = \mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{B}\mathbf{\Gamma}^{-\frac{1}{2}}$ *and* $\mathbf{w} = \mathbf{\Gamma}^{\frac{1}{2}}(\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{d}-\mathbf{c})$.

Proof From Equation (2.6) and Equation (2.13),

$$LOC = \frac{(\mathbf{d}-\mathbf{c})^T[(\mathbf{B}+\mathbf{\Gamma})^{-1}+(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma}(\mathbf{B}+\mathbf{\Gamma})^{-1}](\mathbf{d}-\mathbf{c})}{(\mathbf{d}-\mathbf{c})^T\mathbf{B}^{-1}(\mathbf{d}-\mathbf{c})}. \tag{2.20}$$

We note that $\mathbf{w}^T = (\mathbf{d}-\mathbf{c})^T(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma}^{\frac{1}{2}}$, since $\mathbf{B}$ and $\mathbf{\Gamma}$ are symmetric matrices. An alternative expression for the profit generated under user optimization is obtained by substituting $\mathbf{w} = \mathbf{\Gamma}^{\frac{1}{2}}(\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{d}-\mathbf{c})$ and $\mathbf{G} = \mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{B}\mathbf{\Gamma}^{-\frac{1}{2}}$ into Equation (2.6) as follows:

$$\begin{aligned}
Z_{UO} &= \frac{1}{4}(\mathbf{d}-\mathbf{c})^T[(\mathbf{B}+\mathbf{\Gamma})^{-1}+(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma}(\mathbf{B}+\mathbf{\Gamma})^{-1}](\mathbf{d}-\mathbf{c}),\\
&= \frac{1}{4}(\mathbf{d}-\mathbf{c})^T[\mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{\Gamma}^{\frac{1}{2}}(\mathbf{B}+\mathbf{\Gamma})^{-1}+(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma}^{\frac{1}{2}}\mathbf{\Gamma}^{\frac{1}{2}}(\mathbf{B}+\mathbf{\Gamma})^{-1}](\mathbf{d}-\mathbf{c}),\\
&= \frac{1}{4}(\mathbf{d}-\mathbf{c})^T\mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{w}+\frac{1}{4}\mathbf{w}^T\mathbf{w},\\
&= \frac{1}{4}(\mathbf{d}-\mathbf{c})^T(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma}^{\frac{1}{2}}\mathbf{\Gamma}^{-\frac{1}{2}}(\mathbf{B}+\mathbf{\Gamma})\mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{w}+\frac{1}{4}\mathbf{w}^T\mathbf{w},\\
&= \frac{1}{4}\mathbf{w}^T\mathbf{\Gamma}^{-\frac{1}{2}}(\mathbf{B}+\mathbf{\Gamma})\mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{w}+\frac{1}{4}\mathbf{w}^T\mathbf{w},\\
&= \frac{1}{4}\mathbf{w}^T(\mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{B}\mathbf{\Gamma}^{-\frac{1}{2}}+\mathbf{I})\mathbf{w}+\frac{1}{4}\mathbf{w}^T\mathbf{w},\\
&= \frac{1}{4}\mathbf{w}^T(\mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{B}\mathbf{\Gamma}^{-\frac{1}{2}}+2\mathbf{I})\mathbf{w},\\
&= \frac{1}{4}\mathbf{w}^T(\mathbf{G}+2\mathbf{I})\mathbf{w}. \qquad (2.21)
\end{aligned}$$

Similarly, an expression for the profit under system optimization is obtained by substituting the expressions for $\mathbf{w}$ and $\mathbf{G}$ into Equation (2.13), resulting in

$$\begin{aligned}
Z_{SO} &= \frac{1}{4}(\mathbf{d}-\mathbf{c})^T\mathbf{B}^{-1}(\mathbf{d}-\mathbf{c}),\\
&= \frac{1}{4}(\mathbf{d}-\mathbf{c})^T(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma}^{\frac{1}{2}}\mathbf{\Gamma}^{-\frac{1}{2}}(\mathbf{B}+\mathbf{\Gamma})\mathbf{B}^{-1}(\mathbf{B}+\mathbf{\Gamma})\mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{\Gamma}^{\frac{1}{2}}(\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{d}-\mathbf{c}),\\
&= \frac{1}{4}\mathbf{w}^T\mathbf{\Gamma}^{-\frac{1}{2}}(\mathbf{B}+\mathbf{\Gamma})\mathbf{B}^{-1}(\mathbf{B}+\mathbf{\Gamma})\mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{w},\\
&= \frac{1}{4}\mathbf{w}^T\mathbf{\Gamma}^{-\frac{1}{2}}(\mathbf{I}+\mathbf{\Gamma}\mathbf{B}^{-1})(\mathbf{B}+\mathbf{\Gamma})\mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{w},\\
&= \frac{1}{4}\mathbf{w}^T\mathbf{\Gamma}^{-\frac{1}{2}}(\mathbf{B}+\mathbf{\Gamma}+\mathbf{\Gamma}+\mathbf{\Gamma}\mathbf{B}^{-1}\mathbf{\Gamma})\mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{w},\\
&= \frac{1}{4}\mathbf{w}^T[\mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{B}\mathbf{\Gamma}^{-\frac{1}{2}}+2\mathbf{I}+\mathbf{\Gamma}^{\frac{1}{2}}\mathbf{B}^{-1}\mathbf{\Gamma}^{\frac{1}{2}}]\mathbf{w},\\
&= \frac{1}{4}\mathbf{w}^T[\mathbf{G}+2\mathbf{I}+\mathbf{G}^{-1}]\mathbf{w}. \qquad (2.22)
\end{aligned}$$

It is now clear that the loss of coordination can be expressed as:

$$LOC = \frac{\mathbf{w}^T(\mathbf{G}+2\mathbf{I})\mathbf{w}}{\mathbf{w}^T(\mathbf{G}+\mathbf{G}^{-1}+2\mathbf{I})\mathbf{w}}.$$

∎

2.5.2 An Upper and Lower Bound in Terms of Eigenvalues of G

We first find an upper and lower bound in terms of the maximum and minimum eigenvalue of matrix $\mathbf{G}$ respectively.

Theorem 2.5.2 *Under Assumptions 2.2.1, 2.2.2, 2.2.3 and 2.2.4, the loss of coordination is bounded by*

$$\frac{\lambda_{min}(\mathbf{G})(\lambda_{min}(\mathbf{G})+2)}{(\lambda_{min}(\mathbf{G})+1)^2} \le LOC \le \frac{\lambda_{max}(\mathbf{G})(\lambda_{max}(\mathbf{G})+2)}{(\lambda_{max}(\mathbf{G})+1)^2}.$$

Proof From Equation (2.19),

$$\begin{aligned} LOC &= \frac{\mathbf{w}^T(\mathbf{G}+2\mathbf{I})\mathbf{w}}{\mathbf{w}^T(\mathbf{G}+\mathbf{G}^{-1}+2\mathbf{I})\mathbf{w}}, \\ &= \frac{\mathbf{z}^T\mathbf{z}}{\mathbf{z}^T[\mathbf{I}+(\mathbf{G}+2\mathbf{I})^{-\frac{1}{2}}\mathbf{G}^{-1}(\mathbf{G}+2\mathbf{I})^{-\frac{1}{2}}]\mathbf{z}}, \end{aligned} \tag{2.23}$$

where $\mathbf{z} = (\mathbf{G}+2\mathbf{I})^{\frac{1}{2}}\mathbf{w}$.

Since $(\mathbf{G}+2\mathbf{I})^{-\frac{1}{2}}\mathbf{G}^{-1}(\mathbf{G}+2\mathbf{I})^{-\frac{1}{2}}$ is a similar matrix to matrix $(\mathbf{G}+2\mathbf{I})^{-1}\mathbf{G}^{-1}$, they have the same eigenvalues. Moreover, $(\mathbf{G}+2\mathbf{I})^{-1}\mathbf{G}^{-1}$ is a symmetric matrix and can be unitarily diagonalized as

$$(\mathbf{G}+2\mathbf{I})^{-1}\mathbf{G}^{-1} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^T,$$

where $\mathbf{P}$ is a unitary matrix such that $\mathbf{P}^T\mathbf{P} = \mathbf{I}$, and $\mathbf{\Lambda}$ is a diagonal matrix consisting of the eigenvalues of $(\mathbf{G}+2\mathbf{I})^{-1}\mathbf{G}^{-1}$. Let $\overline{\mathbf{z}} = \mathbf{P}^T\mathbf{z}$. Therefore,

$$\begin{aligned} LOC &= \frac{\mathbf{z}^T\mathbf{z}}{\mathbf{z}^T[\mathbf{I}+\mathbf{P}\mathbf{\Lambda}\mathbf{P}^T]\mathbf{z}}, \\ &= \frac{\overline{\mathbf{z}}^T\overline{\mathbf{z}}}{\overline{\mathbf{z}}^T[\mathbf{I}+\mathbf{\Lambda}]\overline{\mathbf{z}}}. \end{aligned}$$

Let $\lambda_i(\mathbf{G})$ denote the eigenvalues of $\mathbf{G}$. For any vector $\overline{\mathbf{z}}$, an upper and lower bound for the denominator of the above expression is

$$\begin{aligned}
\overline{\mathbf{z}}^T(\mathbf{I}+\mathbf{\Lambda})\overline{\mathbf{z}} &= \sum_{i=1}^{n}\left[1+\lambda_i[(\mathbf{G}+2\mathbf{I})^{-1}\mathbf{G}^{-1}]\right]|\overline{z}_i|^2,\\
&\leq \sum_{i=1}^{n}\left[1+\lambda_{\max}[(\mathbf{G}+2\mathbf{I})^{-1}\mathbf{G}^{-1}]\right]|\overline{z}_i|^2,\\
&= \left[1+\lambda_{\max}[(\mathbf{G}+2\mathbf{I})^{-1}\mathbf{G}^{-1}]\right]\sum_{i=1}^{n}|\overline{z}_i|^2,\\
&= \left[1+\lambda_{\max}[(\mathbf{G}+2\mathbf{I})^{-1}\mathbf{G}^{-1}]\right]\overline{\mathbf{z}}^T\overline{\mathbf{z}},\\
&= \left[1+\lambda_{\max}[\mathbf{G}(\mathbf{G}+2\mathbf{I})]^{-1}\right]\overline{\mathbf{z}}^T\overline{\mathbf{z}}.
\end{aligned}$$

Similarly,

$$\overline{\mathbf{z}}^T(\mathbf{I}+\mathbf{\Lambda})\overline{\mathbf{z}} \geq \left[1+\lambda_{\min}[\mathbf{G}(\mathbf{G}+2\mathbf{I})]^{-1}\right]\overline{\mathbf{z}}^T\overline{\mathbf{z}}.$$

It is clear that $\lambda[\mathbf{G}(\mathbf{G}+2\mathbf{I})]^{-1} = \dfrac{1}{\lambda^2(\mathbf{G})+2\lambda(\mathbf{G})}$. Since $\mathbf{G}$ is a positive definite symmetric matrix, all eigenvalues of $\mathbf{G}$ are positive. Therefore,

$$\begin{aligned}
\lambda_{\max}[\mathbf{G}(\mathbf{G}+2\mathbf{I})]^{-1} &= \frac{1}{\lambda_{\min}[\mathbf{G}(\mathbf{G}+2\mathbf{I})]},\\
&= \frac{1}{\lambda_{\min}^2(\mathbf{G})+2\lambda_{\min}(\mathbf{G})}.
\end{aligned}$$

Similarly,

$$\lambda_{\min}[\mathbf{G}(\mathbf{G}+2\mathbf{I})]^{-1} = \frac{1}{\lambda_{\max}^2(\mathbf{G})+2\lambda_{\max}(\mathbf{G})}.$$

The loss of coordination is therefore lower bounded by

$$
\begin{aligned}
LOC &\geq \frac{1}{1+\lambda_{\max}[\mathbf{G}(\mathbf{G}+2\mathbf{I})]^{-1}}, \\
&= \frac{1}{1+\dfrac{1}{\lambda^2_{\min}(\mathbf{G})+2\lambda_{\min}(\mathbf{G})}}, \\
&= \frac{\lambda_{min}(\mathbf{G})(\lambda_{min}(\mathbf{G})+2)}{(\lambda_{min}(\mathbf{G})+1)^2}.
\end{aligned}
$$

Similarly,

$$
LOC \leq \frac{\lambda_{max}(\mathbf{G})(\lambda_{max}(\mathbf{G})+2)}{(\lambda_{max}(\mathbf{G})+1)^2}.
$$

∎

2.5.3 Lower Bounds in Terms of Quantity Sensitivities

We will now find a lower bound for the loss of coordination in terms of the quantity sensitivity ratios, which are easier to compute.

Theorem 2.5.3 *Under Assumptions 2.2.1, 2.2.2, 2.2.3, 2.2.4, and matrix* $\mathbf{B}$ *(and thus* $\mathbf{G}$*) to be diagonally dominant, the loss of coordination is lower bounded by*

$$
LOC \geq \frac{(1-r_{max}(\mathbf{G}))(3-r_{max}(\mathbf{G}))}{(2-r_{max}(\mathbf{G}))^2} \geq \frac{(1-r_{max}(\mathbf{B}))(3-r_{max}(\mathbf{B}))}{(2-r_{max}(\mathbf{B}))^2}, \qquad (2.24)
$$

where $r_{max}(\mathbf{G})$ *and* $r_{max}(\mathbf{B})$ *are the quantity sensitivity ratios for matrices* $\mathbf{G}$ *and* $\mathbf{B}$ *defined by*

$$
r_{max}(\mathbf{G}) = \max_i \sum_{j\neq i} \frac{|g_{i,j}|}{g_{i,i}}, \quad r_{max}(\mathbf{B}) = \max_i \sum_{j\neq i} \frac{|b_{i,j}|}{b_{i,i}}.
$$

(note: $g_{i,i}=1$*)*

Proof By Gersgorin's Theorem (see Horn and Johnson (1985)), all eigenvalues of $\mathbf{G}$ are located in at least one of the disks:

$$
\{z : |z-g_{i,i}|\} \leq \sum_{j\neq i}^{n} |g_{i,j}|, \quad i=1,2,...,n.
$$

Therefore, we find a lower bound for $\lambda_{min}(\mathbf{G})$ as follows:

$$\lambda_{min}(\mathbf{G}) - g_{i,i} \geq -\sum_{j\neq i}^{n} |g_{i,j}|, \quad i = 1, 2m, ..., n.$$

Since $g_{i,i} = 1$,

$$\begin{aligned} \lambda_{min}(\mathbf{G}) &\geq 1 - \sum_{j\neq i}^{n} |g_{i,j}|, \\ &\geq 1 - r_{max}(\mathbf{G}), \end{aligned}$$

where $r_{max}(\mathbf{G}) = \max_i \sum_{j\neq i} |g_{i,j}|$. Diagonal domimance of matrix $\mathbf{B}$ ensures that $1 - r_{max}(\mathbf{G}) > 0$. Furthermore, $\dfrac{\lambda_{min}(\mathbf{G})(\lambda_{min}(\mathbf{G}) + 2)}{(\lambda_{min}(\mathbf{G}) + 1)^2}$ is increasing in $\lambda_{min}(\mathbf{G})$, for $\lambda_{min}(\mathbf{G}) \geq 0$. Therefore,

$$\begin{aligned} LOC &\geq \frac{\lambda_{min}(\mathbf{G})(\lambda_{min}(\mathbf{G}) + 2)}{(\lambda_{min}(\mathbf{G}) + 1)^2}, \\ &\geq \frac{(1 - r_{max}(\mathbf{G}))(1 - r_{max}(\mathbf{G}) + 2)}{(1 - r_{max}(\mathbf{G}) + 1)^2}, \\ &= \frac{(1 - r_{max}(\mathbf{G}))(3 - r_{max}(\mathbf{G}))}{(2 - r_{max}(\mathbf{G}))^2}. \end{aligned}$$

Consider the matrix $\mathbf{\Gamma}^{-1}\mathbf{B}$, explicitly given by:

$$\mathbf{\Gamma}^{-1}\mathbf{B} = \begin{bmatrix} 1 & -\frac{\beta_{1,2}}{\alpha_1} & \cdots & \cdots & -\frac{\beta_{1,n}}{\alpha_1} \\ -\frac{\beta_{2,1}}{\alpha_2} & 1 & \cdots & \cdots & -\frac{\beta_{2,n}}{\alpha_2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -\frac{\beta_{n-1,1}}{\alpha_{n-1}} & \vdots & \ddots & 1 & -\frac{\beta_{n-1,n}}{\alpha_{n-1}} \\ -\frac{\beta_{n,1}}{\alpha_n} & \cdots & \cdots & -\frac{\beta_{n,n-1}}{\alpha_n} & 1 \end{bmatrix}.$$

Observe that $\mathbf{\Gamma}^{-1}\mathbf{B}$ and $\mathbf{G}$ are similar matrices, since

$$\mathbf{G} = \mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{B}\mathbf{\Gamma}^{-\frac{1}{2}} = \mathbf{\Gamma}^{\frac{1}{2}}[\mathbf{\Gamma}^{-1}\mathbf{B}]\mathbf{\Gamma}^{-\frac{1}{2}}.$$

Hence, $\mathbf{\Gamma}^{-1}\mathbf{B}$ and $\mathbf{G}$ have the same eigenvalues. Using Theorem 2.5.2,

$$\begin{aligned} LOC &\geq \frac{\lambda_{min}(\mathbf{G})(\lambda_{min}(\mathbf{G})+2)}{(\lambda_{min}(\mathbf{G})+1)^2}, \\ &= \frac{\lambda_{min}(\mathbf{\Gamma}^{-1}\mathbf{B})(\lambda_{min}(\mathbf{\Gamma}^{-1}\mathbf{B})+2)}{(\lambda_{min}(\mathbf{\Gamma}^{-1}\mathbf{B})+1)^2}. \end{aligned}$$

Similarly, by Gersgorin's Theorem, all eigenvalues of $\mathbf{\Gamma}^{-1}\mathbf{B}$ are located in at least one of the disks:

$$\{z : |z - (\mathbf{\Gamma}^{-1}\mathbf{B})_{i,i}|\} \leq \sum_{j\neq i}^{n} |(\mathbf{\Gamma}^{-1}\mathbf{B})_{i,j}|, \quad i = 1, 2, ..., n.$$

Therefore, a lower bound for $\lambda_{min}(\mathbf{\Gamma}^{-1}\mathbf{B})$ is

$$\begin{aligned} \lambda_{min}(\mathbf{\Gamma}^{-1}\mathbf{B}) &\geq 1 - \sum_{j\neq i}^{n} |(\mathbf{\Gamma}^{-1}\mathbf{B})_{i,j}|, \\ &= 1 - \sum_{j\neq i}^{n} \frac{\beta_{i,j}}{\alpha_i}, \\ &\geq 1 - \max_i \sum_{j\neq i}^{n} \frac{\beta_{i,j}}{\alpha_i}, \\ &= 1 - r_{max}(\mathbf{B}) \end{aligned}$$

Diagonal domimance of $\mathbf{B}$ ensures that $1 - r_{max}(\mathbf{B}) > 0$. We now have a lower bound for the loss of coordination in terms of the quantity sensitivity computed directly from $\mathbf{B}$, given by:

$$\begin{aligned} LOC &\geq \frac{(1 - r_{max}(\mathbf{B}))(1 - r_{max}(\mathbf{B}) + 2)}{(1 - r_{max}(\mathbf{B}) + 1)^2}, \\ &= \frac{(1 - r_{max}(\mathbf{B}))(3 - r_{max}(\mathbf{B}))}{(2 - r_{max}(\mathbf{B}))^2}. \end{aligned}$$

∎

Remark The additional assumption, requiring matrix $\mathbf{B}$ to be diagonally dominant, implies that a retailer's quantity policy has a higher effect on its market clearing price than the total effect of the quantity policies of all other retailers. This assumption is valid in markets where market prices decrease with when every retailer increases supply by one unit.

2.6 Tightness of Bound

We analyze the tightness of the lower bounds in terms of the minimum eigenvalue of the quantity sensitivity matrix $\mathbf{B}$, and the bound in terms of the quantity sensitivity ratio by varying the number of retailers, n. We generate 10000 random instances of matrix $\mathbf{B}$, which satisfies the assumptions in Section 2.2.1, for each n (n varying from 2 to 20). We then obtain the averages of the loss of coordination and their bounds from these random instances. In these simulations, we consider two scenarios for vector $\mathbf{d}$:

1. $\mathbf{d} = \mathbf{c} + \mathbf{r}$, where $\mathbf{r}$ is a random vector.

2. $\mathbf{d} = \mathbf{c} + \mathbf{k}$, where $\mathbf{k}$ is a constant vector of ones.

Let the lower bound in terms of $\lambda_{min}(\mathbf{G})$ and $r_{max}(\mathbf{B})$ be denoted by $LOC(\lambda_{min}(\mathbf{G}))$ and $LOC(r_{max}(\mathbf{B}))$ respectively. The results of the simulations are shown in Figure 2-1 and 2-2.

Discussion

1. We observe that the average actual loss of total profit in the supply chain is always less than 30%. This shows that the uncoordinated supply chain is 'fairly' efficient, with the average loss of total profits in the supply chain due to competition consistently below 30%. The efficiency of the uncoordinated supply chain is better when $\mathbf{d} = \mathbf{c} + \mathbf{k}$, where $\mathbf{k}$ is a constant vector of ones. In this scenario, the loss of total profits due to competition is only about 20%.

2. The bound in terms of $\lambda_{min}(\mathbf{G})$ is much tighter than the one in terms of quantity sensitivity, as expected from the derivations of the lower bounds. For example, when $n = 20$, the actual LOC is 0.78 (i.e., the loss of profit is 22%). The bound

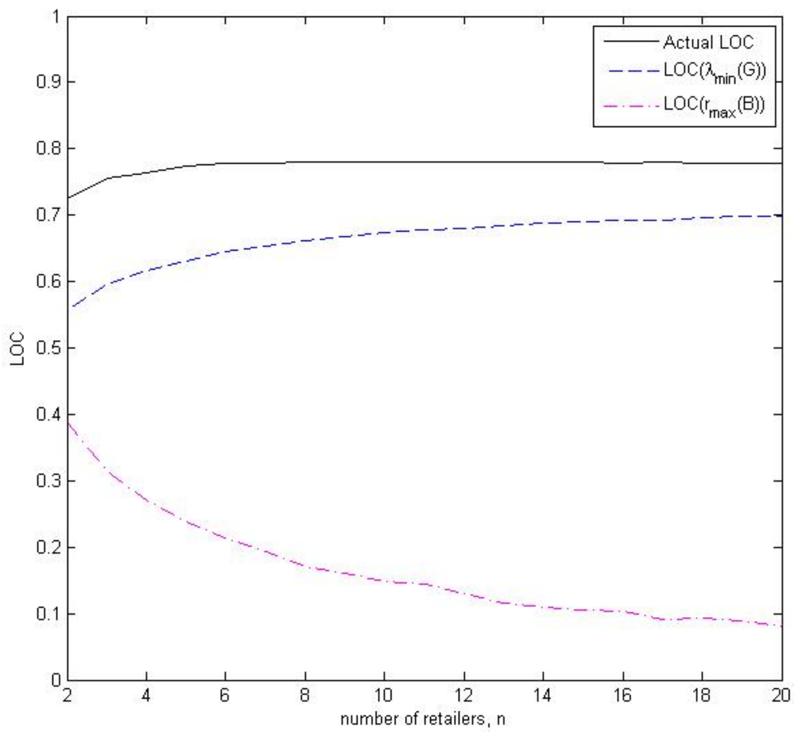

Figure 2-1: Lower bounds with varying number of retailers, when $\mathbf{d} = \mathbf{c} + \mathbf{r}$, where $\mathbf{r}$ is a random vector.

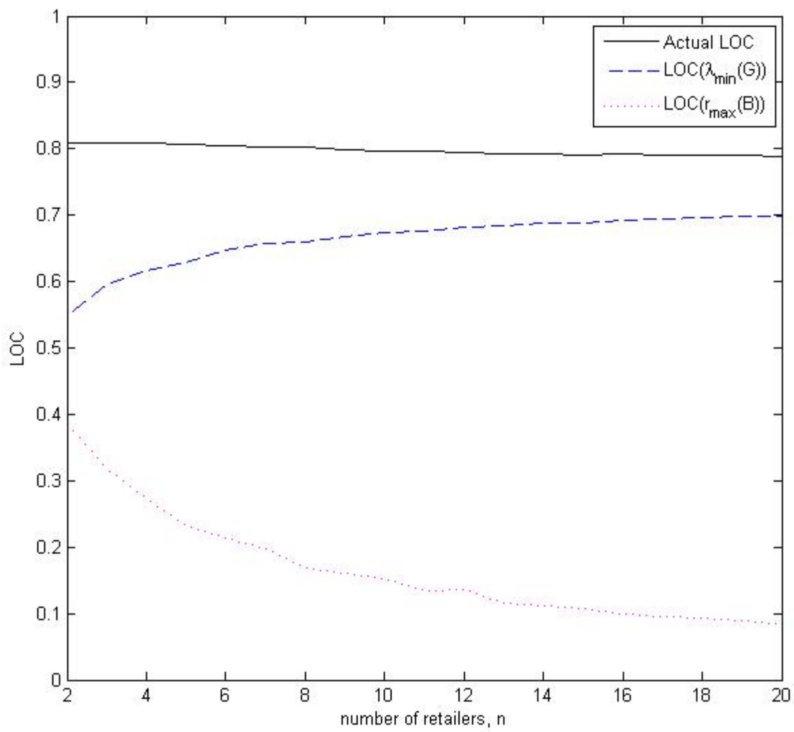

Figure 2-2: Lower bounds with varying number of retailers, when $\mathbf{d} = \mathbf{c} + \mathbf{k}$, where $\mathbf{k}$ is a vector of ones.

in terms of the minimum eigenvalue gives a lower bound of 0.70. Notice the bound in terms of quantity sensitivity gives a worse lower bound of 0.09.

3. The bounds are tighter when $\mathbf{d} = \mathbf{c} + \mathbf{r}$, where $\mathbf{r}$ is a random vector, due to a lower average loss of coordination.

2.7 Loss of Coordination under Uniform Demand

In this section, we analyze the loss of coordination in a symmetric setting under the uniform demand model without quality differences among products from different sellers.

2.7.1 Model Description

In this setting, all retailers encounter identical quantity sensitivities and the same demand function for all their products.
The following assumptions, in addition to those in Section 2.2.1, will be imposed throughout this section.

Assumption 2.7.1 *The quantity sensitivity is identical for all retailers. That is,* $\alpha_i = \alpha$, $\beta_i = \beta$, $p_i = d_i - \alpha q_i + \beta \mathbf{q_{-i}}$ *for all* $i = 1, 2, ..., n$. *Without loss of generality, we set* $\beta = 1$.

Assumption 2.7.2 *There is no quality differences between retailers. Moreover, the supplier incurs the same cost per unit quantity ordered by each retailer. That is,* $\mathbf{d} = (d, d, ..., d)^T$ *and* $\mathbf{c} = (c, c, ..., c)^T$.

Assumption 2.7.3 *The market clearing prices under zero production is at least as high as the per-unit costs incurred by the supplier, which must be non-negative. That is,* $\mathbf{d} \geq \mathbf{c} \geq 0$.

The vector $\mathbf{d}$ indicates the base demand prices (i.e., prices when quantities are zero) for the products by each retailer. If they are lower than the production costs, we can assume that the product is removed from the market in order for the firms to be profitable.

Recall that we are dealing with subsititute products in a quantity competition. Therefore, $\overline{\mathbf{B}}$ is an M-matrix in the price-demand relationship $\mathbf{q} = \overline{\mathbf{d}} - \overline{\mathbf{B}}\mathbf{p}$. By Assumption 2.7.1, matrix $\overline{\mathbf{B}}$ is

$$\overline{\mathbf{B}} = k \begin{bmatrix} \alpha' & -1 & \cdots & \cdots & -1 \\ -1 & \alpha' & \cdots & \cdots & -1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & \vdots & \ddots & \alpha' & -1 \\ -1 & \cdots & \cdots & -1 & \alpha' \end{bmatrix},$$

The price demand relationship can be expressed as $\mathbf{p} = \mathbf{d} - \mathbf{B}\mathbf{q}$, where $\mathbf{B} = \overline{\mathbf{B}}^{-1}$ given by

$$\mathbf{B} = \begin{bmatrix} \alpha & 1 & \cdots & \cdots & 1 \\ 1 & \alpha & \cdots & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \vdots & \ddots & \alpha & 1 \\ 1 & \cdots & \cdots & 1 & \alpha \end{bmatrix}.$$

2.7.2 User Optimization

We will present the equilibrium prices, quantities and profits under user optimization.

Proposition 2.7.4 *Under Assumption 2.7.1, 2.7.2 and 2.7.3, the equilibrium total profit generated under uniform demand in user optimization is*

$$Z_{UO} = \frac{3\alpha - 1 + n}{4(2\alpha - 1 + n)^2} n(d-c)^2, \tag{2.25}$$

with equilibrium quantities, market clearing prices and wholesale prices given respec-

tively by the vectors

$$\mathbf{q_{UO}} = \frac{d-c}{4\alpha - 2 + 2n}\mathbf{e}, \tag{2.26}$$

$$\mathbf{p_{UO}} = \frac{(3\alpha + 1 - n)d + (\alpha + 1 - n)c}{4\alpha + 2 - 2n}\mathbf{e}, \tag{2.27}$$

$$\mathbf{w_{UO}} = \frac{d+c}{2}\mathbf{e}.$$

Proof First, observe that $\mathbf{w_{UO}} = \frac{d+c}{2}\mathbf{e}$ follows directly from Equation (2.11). Also note that we can express the matrices $\mathbf{B}$ and $\mathbf{B}+\mathbf{\Gamma}$ by

$$\mathbf{B} = (\alpha - 1)\mathbf{I} + \mathbf{H},$$

$$\mathbf{B} + \mathbf{\Gamma} = (2\alpha - 1)\mathbf{I} + \mathbf{H},$$

where

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & \cdots & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \vdots & \ddots & 1 & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{bmatrix}.$$

We rewrite matrix $(\mathbf{B}+\mathbf{\Gamma})^{-1}$ as follows:

$$\begin{aligned} (\mathbf{B}+\mathbf{\Gamma})^{-1} &= [(2\alpha - 1)\mathbf{I} + \mathbf{H}]^{-1}, \\ &= \frac{1}{2\alpha - 1}[\mathbf{I} + \frac{1}{2\alpha - 1}\mathbf{H}]^{-1}, \\ &= \frac{1}{2\alpha - 1}[\mathbf{I} - \frac{1}{2\alpha + 1}\mathbf{H} + (\frac{1}{2\alpha + 1}\mathbf{H})^2 - (\frac{1}{2\alpha + 1}\mathbf{H})^3 + ...]. \end{aligned}$$

Since $\mathbf{H}^k = n^{k-1}\mathbf{H}$, it follows that

$$
\begin{aligned}
(\mathbf{B}+\mathbf{\Gamma})^{-1} &= \frac{1}{2\alpha-1}[\mathbf{I} - \frac{1}{2\alpha-1}\mathbf{H} + \frac{n}{(2\alpha-1)^2}\mathbf{H} - \frac{n^2}{(2\alpha-1)^3}\mathbf{H}...], \\
&= \frac{1}{2\alpha-1}[\mathbf{I} + \frac{\frac{-1}{2\alpha-1}}{1-\frac{-n}{2\alpha-1}}\mathbf{H}], \\
&= \frac{1}{2\alpha-1}[\mathbf{I} - \frac{1}{2\alpha-1+n}\mathbf{H}].
\end{aligned}
$$

From Equation (2.7),

$$
\begin{aligned}
\mathbf{q_{UO}} &= \frac{1}{2}(\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{d}-\mathbf{c}), \\
&= \frac{1}{2}\frac{1}{2\alpha-1}[\mathbf{I} - \frac{1}{2\alpha-1+n}\mathbf{H}](\mathbf{d}-\mathbf{c}), \\
&= \frac{1}{4\alpha-2}(1-\frac{n}{2\alpha-1+n})(d-c)\mathbf{e}, \\
&= \frac{d-c}{4\alpha-2+2n}\mathbf{e}.
\end{aligned}
$$

The equilibrium market clearing prices follows directly from the equilibrium quantities and price demand relationship assumed in Assumption 2.2.1. From the affine price demand relationship and Equation (2.26),

$$
\begin{aligned}
\mathbf{p_{UO}} &= \mathbf{d} - \mathbf{B}\mathbf{q_{UO}}, \\
&= \mathbf{d} - [(\alpha-1)\mathbf{I}+\mathbf{H}]\frac{d-c}{4\alpha-2+2n}\mathbf{e}, \\
&= \left[d-(\alpha-1+n)\frac{d-c}{4\alpha-2+2n}\right]\mathbf{e}, \\
&= \frac{(3\alpha-1+n)d+(\alpha-1+n)c}{4\alpha-2+2n}\mathbf{e}.
\end{aligned}
$$

Therefore, the equilibrium total profit under user optimization is

$$\begin{aligned}
Z_{UO} &= (\mathbf{p_{UO}} - \mathbf{c})^T \mathbf{q_{UO}}, \\
&= n\left(\frac{(3\alpha - 1 + n)d + (\alpha - 1 + n)c}{4\alpha - 2 + 2n} - c\right)\frac{d - c}{4\alpha - 2 + 2n}, \\
&= \frac{n(d-c)}{(4\alpha - 2 + 2n)^2}(3\alpha - 1 + n)(d - c), \\
&= \frac{3\alpha - 1 + n}{4(2\alpha - 1 + n)^2} n(d-c)^2.
\end{aligned}$$

$\blacksquare$

2.7.3 System Optimization

We will present the optimal prices, quantities and profits under system optimization.

Proposition 2.7.5 *Under Assumption 2.7.1, 2.7.2 and 2.7.3, the equilibrium total profit generated under uniform demand in system optimization is*

$$Z_{SO} = \frac{n(d-c)^2}{4\alpha - 4 + 4n}, \tag{2.28}$$

with optimal production quantities and market clearing prices given by the vectors

$$\mathbf{q_{SO}} = \frac{d - c}{2\alpha - 2 + 2n}\mathbf{e},$$

$$\mathbf{p_{SO}} = \frac{d + c}{2}\mathbf{e}.$$

Proof The proof is similar to the one in Theorem 2.7.4. We write $\mathbf{B}^{-1}$ as follows:

$$
\begin{aligned}
\mathbf{B}^{-1} &= [(\alpha-1)\mathbf{I}+\mathbf{H}]^{-1}, \\
&= \frac{1}{\alpha-1}[\mathbf{I}+\frac{1}{\alpha-1}\mathbf{H}]^{-1}, \\
&= \frac{1}{\alpha-1}[\mathbf{I}-\frac{1}{\alpha-1}\mathbf{H}+(\frac{1}{\alpha-1}\mathbf{H})^2-(\frac{1}{\alpha-1}\mathbf{H})^3+...]. \\
&= \frac{1}{\alpha-1}[\mathbf{I}+\frac{-1}{\alpha-1}\mathbf{H}+\frac{n}{(\alpha-1)^2}\mathbf{H}-\frac{n^2}{(\alpha-1)^3}\mathbf{H}...], \\
&= \frac{1}{\alpha-1}[\mathbf{I}+\frac{\frac{-1}{\alpha-1}}{1-\frac{-n}{\alpha-1}}\mathbf{H}], \\
&= \frac{1}{\alpha-1}[\mathbf{I}+\frac{1}{\alpha-1+n}\mathbf{H}].
\end{aligned}
$$

From Equation (2.18),

$$
\begin{aligned}
\mathbf{q_{SO}} &= \frac{1}{2}\mathbf{B}^{-1}(\mathbf{d}-\mathbf{c}), \\
&= \frac{1}{2}\frac{1}{\alpha-1}[\mathbf{I}+\frac{1}{\alpha-1+n}\mathbf{H}](\mathbf{d}-\mathbf{c}), \\
&= \frac{1}{2\alpha-2}(1+\frac{n}{\alpha-1+n})(d-c)\mathbf{e}, \\
&= \frac{d-c}{2\alpha-2+2n}\mathbf{e}.
\end{aligned}
$$

From Equation (2.13),

$$
\begin{aligned}
Z_{SO} &= \frac{1}{4}(\mathbf{d}-\mathbf{c})^T\mathbf{B}^{-1}(\mathbf{d}-\mathbf{c}), \\
&= \frac{1}{4}(\mathbf{d}-\mathbf{c})^T\frac{1}{\alpha-1}[\mathbf{I}+\frac{1}{\alpha-1+n}\mathbf{H}](\mathbf{d}-\mathbf{c}), \\
&= \frac{1}{4}(\mathbf{d}-\mathbf{c})^T\frac{1}{\alpha-1}(1+\frac{n}{\alpha-1+n})(d-c)\mathbf{e}, \\
&= \frac{1}{4}\frac{d-c}{\alpha-1+n}(\mathbf{d}-\mathbf{c})^T\mathbf{e}, \\
&= \frac{1}{4}\frac{d-c}{\alpha-1+n}(\mathbf{d}-\mathbf{c})^T\mathbf{e}, \\
&= \frac{n(d-c)^2}{4\alpha-4+4n}.
\end{aligned}
$$

Following directly from Equation (2.14),

$$\mathbf{p_{SO}} = \frac{d+c}{2}\mathbf{e}.$$

$\blacksquare$

Remark The optimal profit generated under system optimization is independent of the wholesale price, $\mathbf{w_{SO}}$.

2.7.4 Analysis of Loss of Coordination under Uniform Demand

We will study the efficency of the system under uniform demand by analyzing the loss of coordination. We will first give the expression for the loss of coordination in the following theorem:

Theorem 2.7.6 *Under Assumption 2.7.1, 2.7.2 and 2.7.3, the loss of coordination under a uniform demand function in a Cournot competition with substitute products is*

$$LOC = \frac{(3+r)(1+r)}{(2+r)^2},$$

where $r = \dfrac{n-1}{\alpha}$.

Proof From Equation (2.25) and Equation (2.28), the loss of coordination is

$$\begin{aligned}
LOC &= \frac{Z_{UO}}{Z_{SO}}, \\
&= \frac{3\alpha - 1 + n}{4(2\alpha - 1 + n)^2}(4\alpha - 4 + 4n), \\
&= \frac{(3\alpha - 1 + n)(\alpha - 1 + n)}{(2\alpha - 1 + n)^2}, \\
&= \frac{(3 + \frac{n-1}{\alpha})(1 + \frac{n-1}{\alpha})}{(2 + \frac{n-1}{\alpha})^2}, \\
&= \frac{(3+r)(1+r)}{(2+r)^2}.
\end{aligned}$$

$\blacksquare$

Remark Since $\lambda_{max}(\mathbf{G}) = 1 + r$ for uniform demand (i.e., symmetric retailers), the LOC upper bound in Theorem 2.5.2 is tight (i.e., it is achieved for symmetric retailers under uniform demand).

Proposition 2.7.7 *Under Assumption 2.2.4, 2.7.1, 2.7.2 and 2.7.3, the loss of coordination under uniform demand in a quantity competition with substitute products is bounded by*

$$\frac{3}{4} \leq LOC \leq \frac{n(n+2)}{(n+1)^2} \tag{2.29}$$

Proof We will first determine the range of α under the diagonal dominance of $\overline{\mathbf{B}}$ in Assumption 2.2.4. From

$$\overline{\mathbf{B}} = k \begin{bmatrix} \alpha' & -1 & \cdots & \cdots & -1 \\ -1 & \alpha' & \cdots & \cdots & -1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & \vdots & \ddots & \alpha' & -1 \\ -1 & \cdots & \cdots & -1 & \alpha' \end{bmatrix},$$

where $k = (\alpha' + 1)(\alpha' + 1 - n)$, we compute $\mathbf{B}$ by taking its inverse. Therefore,

$$\mathbf{B} = \begin{bmatrix} \alpha & 1 & \cdots & \cdots & 1 \\ 1 & \alpha & \cdots & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \vdots & \ddots & \alpha & 1 \\ 1 & \cdots & \cdots & 1 & \alpha \end{bmatrix},$$

where $\alpha = \alpha' + 2 - n$. By Assumption 2.2.4, we have $\alpha' \geq n - 1$, which leads to

$$\alpha \geq 1.$$

Therefore, the range of r is given by

$$0 < r = \frac{n-1}{\alpha} \leq n - 1.$$

Since the loss of coordination is increasing with respect to r, the maximum and the minimum loss of coordination is attained by the maximum and minimum r respectively. Therefore,

$$\frac{3}{4} < LOC = \frac{(3+r)(1+r)}{(2+r)^2} \leq \frac{n(n+2)}{(n+1)^2}.$$

■

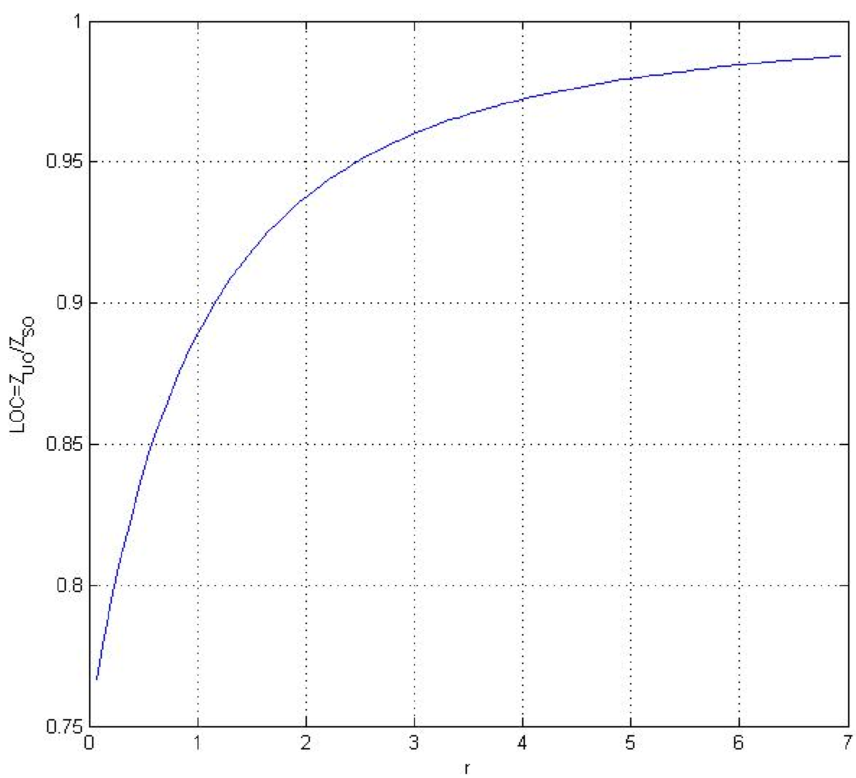

Figure 2-3: The loss of coordination in a Cournot competition with substitute products and uniform demand.

Discussions

We perform two sets of simulations. First, we determine the loss of coordination in a supply chain of eight retailers by varying the values of r under the uniform demand. Figure 2-3 shows the result of this set of simulations. We also compare the loss of coordination when $\mathbf{d} - \mathbf{c}$ is a random vector with each component having

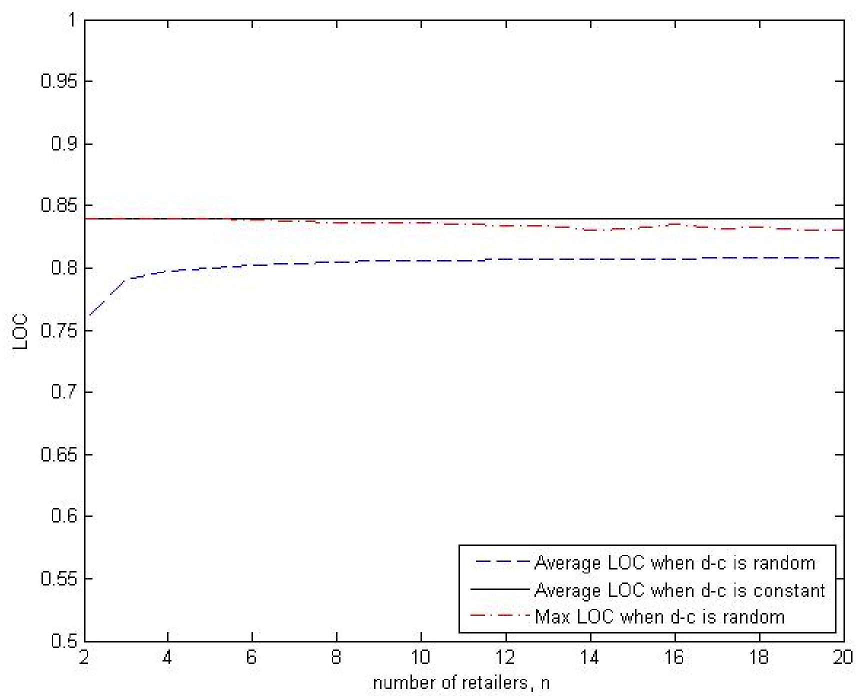

Figure 2-4: Comparing the loss of coordination when $\mathbf{d}-\mathbf{c}$ is a constant and when it is random.

mean of one, satisfying Assumption 2.2.5, and when $\mathbf{d} - \mathbf{c}$ is the constant vector of ones. We generate 10,000 instances of $\mathbf{d}$, for varying number of retailers from 2 to 20, for a fixed quantity sensitivity ratio of $r = 0.5$, under the scenario when all retailers encounter identical quantity sensitivities. We therefore use matrix $\mathbf{B}$ as shown in Section 2.7.1. Figure 2-4 shows the result of this simulation. There are several observations regarding the loss of coordination as illustrated in the figures.

1. The supply chain is more efficient when retailers are under intense competition, measured by the values of r. When r is large, the competitors' quantity policies has a greater effect on a retailer's market clearing price, suggesting a greater intensity of competition. There can be close to no loss in efficiency when there is a 'reasonable' number of retailers under intense competition. For example, the loss of efficiency due to intense competition (i.e., when $r = n - 1$) is 1.23%, 2.04% and 4% in the presence of eight, six and four retailers respectively.

2. The loss of coordination, LOC, is lower bounded by 0.75, and is lowest when r is small, which happens when α is large (for example, in the presence of six retailers, $LOC = 0.762$ when $\alpha = 100$, $r = 0.05$). This results in the retailers becoming more independent of one another. Therefore, the resultant supply chain acts as if there were several independent supply chains of one supplier and one retailer. The loss of coordination, as a result, is given by the loss of coordination under the configuration of one supplier and one retailer, which is $\frac{3}{4}$.

3. The loss of coordination increases with n, and is 'almost' coordinated when $n \to \infty$. This is consistent with Adida and DeMiguel (2009), which explains that competition between retailers intensifies when the number of retailers increases, and they ultimately become price takers. Meanwhile, the market power of the supplier increases, resulting in the decentralized supply chain converging to one that is centralized.

4. The bound of the loss of coordination given in Proposition 2.7.7 is tight, as illustrated by Figure 2-3.

5. The loss of coordination is a concave function with respect to r. As such, the loss of coordination is high for a wide range of values of r. For instance, the loss of coordination stays above 0.9 for $1.2 \leq r \leq 7$, which is more than 82% of the data instances of r.

6. The efficiency of the supply chain is higher when $\mathbf{d} - \mathbf{c}$ is a constant vector of ones. This imply that there is no quality differences among the retailers. Furthermore, this gives an upper bound for the LOC compared to the case when $\mathbf{d}$ is a random vector, since it is consistently higher the maximum of the LOC when $\mathbf{d}$ is random, for all data instances and for all number of retailers tested.

Chapter 3

Bertrand Competition with Substitute Products

3.1 Overview and Main Contributions

In this chapter, we analyze the loss of coordination in a two-tier single-supplier, multi-retailer supply chain setting. The supply chain we consider is a Stackelberg game where the supplier is the leader and the retailers are the followers. The retailers compete in an oligopoly market through deciding prices (Bertrand competition) of substitute products. Our model considers an affine demand price relation. This arises naturally from a quasilinear consumer utility function. As a special case, we also consider a uniform demand function, when all retailers encounter identical demand (i.e., have the same price sensitivities for all products). The demand function represents the consumers in an aggregate format and depends only on the prices set by the retailers.

We evaluate the loss of coordination to measure the efficiency of the supply chain under competition, computed as the ratio of the total profit (that is, the total supplier's and retailers' profit) generated under competition (user optimization) and under coordination (system optimization). We then propose lower bounds for this loss of coordination to quantify the efficiency of the supply chain under competition. The lower bounds are in terms of the eigenvalues of the demand sensitivity matrix, or the demand sensitivities. In addition, we conduct simulations which further indicate that the maximum loss due to competition of the supply chain is no more than 25%, and the average loss is less than 15%. This implies that the competitive (uncoordinated) supply chain is in fact 'fairly' efficient. Moreover, theoretical and simulation results both indicate that under uniform demand, the supply chain can be 'almost' coordinated when demand is inelastic (i.e., demand is not 'significantly' affected by prices).

The structure of the remainder of this chapter is as follows. Section 3.2 provides the groundwork for this chapter. Subsection 3.2.1 gives the notations, definitions and assumptions imposed in our analysis. We discuss the rationale and validity of these assumptions. In Subsection 3.2.2, we describe the model and review the central concepts of Nash equilibrium, user optimum, system optimum and the loss of coordination. Section 3.3 presents the equilibrium wholesale prices, selling prices, demand, and total profits under user optimization, when individual market participants maximize their own profits. In Section 3.4, we derive the optimal selling prices, demand and total profits achieved under system optimization, when a central authority is coordinating decisions.

Section 3.5 presents the most important findings in this chapter - the loss of coordination in terms of the quantity sensitivity matrix in Subsection 3.5.1, and presents lower bounds for this loss of coordination. We present two lower bounds in Subsection 3.5.2. One lower bound is in terms of the minimum eigenvalue of the price sensitivity matrix, and one lower bound is in terms of the price sensitivity ratio, which is easier to compute. Simulations are performed in Section 3.6 to evaluate and compare the tightness of these bounds. Numerical results from these simulations also indicate that the average loss due to competition in the supply chain is no more than 15%. Finally, in Section 3.7, we analyze the loss of coordination under the uniform demand model where all retailers encounter identical price sensitivities and experience the same demand function for all their products. Results under the uniform demand model indicate that it is possible to attain close to no loss in efficiency when demand in perfectly inelastic.

3.2 Preliminaries

We consider a two tier single-supplier, multi-retailer supply chain producing differentiated substitute products under an affine price demand relation competiting in a Bertrand (price) oligopoly market, where retailers compete by deciding the selling prices to charge the consumers. We will first list the associated notations and assumptions in Section 3.2.1, and give more specific details of the model in Section 3.2.2.

3.2.1 Assumptions and Notations

In this supply chain with a single supplier and n retailers, we denote the order quantity of retailer i ($i = 1, 2, ..., n$) by q_i and let vector $\mathbf{q} = (q_1, ..., q_n)^T$. Similarly, let vectors $\mathbf{d}$, $\mathbf{c}$, $\mathbf{w}$ and $\mathbf{p}$ be the respective vectors for the demand under zero selling price, the costs per unit order incurred by the supplier, the wholesale prices charged by the supplier and the selling prices. Let Z_{R_i} be the profit of retailer i, and Z_S be the supplier's profit.

Let the equilibrium wholesale prices, selling prices, order quantities and total profits under competition (user optimization) be denoted by $\mathbf{w_{UO}}$, $\mathbf{p_{UO}}$, $\mathbf{q_{UO}}$ and Z_{UO} respectively. Let the optimal selling prices, order quantities and total profits under coordination (system optimization) be denoted by $\mathbf{p_{SO}}$, $\mathbf{q_{SO}}$ and Z_{SO} respectively.

Our analysis is restricted to models that satisfy the following assumptions:

Assumption 3.2.1 *The price demand relationship is affine and deterministic.*

This imply that $\mathbf{q}(\mathbf{p}) = \mathbf{d} - \mathbf{B}\mathbf{p}$, where $\mathbf{B}$ is the price sensitivity matrix. Affine demand functions are common in the pricing literature. Such a model arises naturally from a quasilinear utility function of a representative consumer. This model has been used by many researchers such as Carr et al. (1999), Berstein and Federgruen (2003), Allon and Federgruen (2006, 2007). In this thesis, we remove the effects of stochasticity of demand in order to isolate the effects of competition.

Assumption 3.2.2 *The market demand when products are priced at cost must be non-negative. That is,* $\mathbf{d} \geq \mathbf{B}\mathbf{c}$.

Otherwise, we can assume that the product is removed from the market in order for the firms to be profitable.

Assumption 3.2.3 *The marginal costs of production and demand when prices are zero are non-negative. That is,* $\mathbf{d} \geq 0$ *and* $\mathbf{c} \geq 0$*.*

If the demand when prices are zero are negative, we can assume that the product is removed from the market in order for the firms to be profitable.

Assumption 3.2.4 *The following holds at equilibrium under user optimization:*

$$\mathbf{B}(\mathbf{B} + \mathbf{\Gamma})^{-1}\mathbf{\Gamma}(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c}) \geq 0,$$

As we will see later in this chapter, this implies that all retailers' demand are non-negative. This requires the market demand at zero selling price, $\mathbf{d}$, and the vector of costs incurred by the supplier, $\mathbf{c}$, to be such that it is profitable to supply and sell a non-negative amount of the product. This assumption is valid because products which do not satisfy this requirement are not profitable to produce and naturally do not exist in the market at equilibrium. See also Adida and DeMiguel (2009) who also imposes and discusses this assumption.

Note that when demand is uniform (see Section 3.7 for definition), this assumption is equivalent to $d \geq (\alpha + n - 1)c$. This follows under the assumption that prices at cost are non-negative (i.e., Assumption 3.2.2).

Assumption 3.2.5 *The price sensitivity matrix,* $\mathbf{B}$*, is a symmetric matrix.*

This assumption implies that the cross-effects of the retailers' prices on each other are symmetric. This model arises naturally when a representative consumer maximizes a quasilinear utility function.

Assumption 3.2.6 *Matrix* $\mathbf{B}$ *has positive diagonals and non-negative off-diagonals.*

This is a natural consequence of a market with substitute products. Increasing a retailer's selling price has a negative effect on its own market demand, but a non-negative effect on other retailers' demand.

Assumption 3.2.7 $\mathbf{B}$ *is a column-diagonally dominant matrix.*

This implies that a retailer's pricing policy has a higher effect on its market demand than the total effect of the pricing policies of all other retailers. This is applicable to markets where the total market demand decreases with an increase in selling price of one retailer.

Let $\mathbf{B}$ be the following matrix:

$$\mathbf{B} = \begin{bmatrix} \alpha_1 & -\beta_{1,2} & \cdots & \cdots & -\beta_{1,n} \\ -\beta_{2,1} & \alpha_2 & \cdots & \cdots & -\beta_{2,n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -\beta_{n-1,1} & \vdots & \ddots & \alpha_{n-1} & -\beta_{n-1,n} \\ -\beta_{n,1} & \cdots & \cdots & -\beta_{n,n-1} & \alpha_n \end{bmatrix},$$

and let $\mathbf{\Gamma}$ be a diagonal matrix consisting only of the diagonals of matrix $\mathbf{B}$.

Remark Assumption 3.2.6 requires $\alpha_i > 0$ and $\beta_{i,j} \geq 0$ for all i, j. Assumption 3.2.7 requires $|\alpha_i| \geq \sum_{i \neq j} |\beta_{i,j}|$ for all i, j. These assumptions imply that $\mathbf{B}$ is an M-matrix, defined in Section 2.2.2.

Definition *(Price sensitivity ratio)* The price sensitivity ratio, $r_i(\mathbf{B})$, for retailer i is obtained from the price sensitivity matrix $\mathbf{B}$. It is defined as

$$r_i(\mathbf{B}) = \sum_{i \neq j} \frac{\beta_{i,j}}{\alpha_i}.$$

For other definitions that will be used in this chapter, we refer the reader to Section 2.2.2.

3.2.2 Model Description

We consider a two tier single-supplier, multi-retailer supply chain producing differentiated substitute products under an affine demand function.

The sequence of events is as follows. The supplier is a Stackelberg leader who first proposes a wholesale price to each of the retailers. After receiving the wholesale price, each retailer makes a decision on their own selling price, and specifies to the supplier his/her respective order quantity to fully satisfy the market demand. Upon receiving the order quantities, the supplier begins production and delivers items to each retailer at costs incurred by the supplier.

In a Bertrand (price) oligopoly market, retailer compete by deciding the selling prices to charge the consumers, with the market demand determined as functions of the selling price through the price demand relationship.

Under user optimization, the supplier maximizes her profit by deciding the wholesale prices as a best response to the anticipated equilibrium order quantities by the retailers. The retailers decide on the selling prices to charge the consumers in response to the supplier's pricing policies. The supplier and each retailer is assumed to be rational and selfish, optimizing profits only for themselves. Nash Equilibrium is reached when no single retailer can increase its profit by unilaterally changing its policy.

For each retailer i, given the supplier's equilibrium wholesale price, w_i obtained from the vector $\mathbf{w_{UO}}$, and competitors' equilibrium selling prices given by the vector $\mathbf{p_{UO,-i}}$, the retailer's best response quantity policy is obtained by solving the optimization problem $\mathbf{UO_{R_i}}$ described as follows:

$$\begin{aligned} \mathbf{UO_{R_i}}: \quad & \max_{p_i} \quad q_i(p_i, \mathbf{p_{UO,-i}}).(p_i - w_i), \\ & \text{s.t.} \quad q_i \geq 0, p_i \geq w_i \end{aligned} \tag{3.1}$$

The equilibrium wholesale price, w_i, for retailer i in the above problem is the solution to the supplier's optimization problem. The supplier maximizes profit by deciding the wholesale price vector, $\mathbf{w_{UO}}$, given the retailers' equilibrium order quantities obtained from vector $\mathbf{q_{UO}}$. This optimization problem, $\mathbf{UO_S}$, is described as follows:

$$
\begin{aligned}
\mathbf{UO_S}: \quad \max_{\mathbf{w}} \quad & \sum_{i=1}^{n} (w_i - c_i).q_i(w_i, \mathbf{w_{UO,-i}}), \\
\text{s.t.} \quad & q_i \geq 0, w_i \geq c_i \quad \text{for all} \quad i = 1, ..., n.
\end{aligned}
\tag{3.2}
$$

Let $Z_{R_i}^{UO}$ denote the profit of retailer i obtained from Optimization Problem (3.1), and Z_S^{UO} be the profit of the supplier obtained from Optimization Problem (3.2). The total profit under user optimization, Z_{UO}, is the sum of the profits of all the retailers and the supplier given by

$$
Z_{UO} = Z_S^{UO} + \sum_{i=1}^{n} Z_{R_i}^{UO}.
$$

Under system optimization, a central authority is coordinating all decisions, optimizing the total profit of the supplier and all retailers. The central authority makes decisions on all production quantities and forces the supplier and all retailers to comply. Coordination is attained by solving the following optimization problem, which determines the production quantities that maximize the total supply chain profit of the supplier and all retailers.

$$
\begin{aligned}
\textbf{SO:} \quad \max_{\mathbf{q_{SO}}} \quad & Z_S + \sum_{i=1}^{n} Z_{R_i}, \\
\text{s.t.} \quad & \mathbf{q_{SO}} \geq 0, \mathbf{p_{SO}} \geq \mathbf{c}
\end{aligned}
\tag{3.3}
$$

Let Z_{SO} denote the optimal total profit obtained by solving the above optimization problem.

The loss of coordination, LOC, measures the loss of the total supply chain profit under competition, computed as the ratio of the total profit generated under user optimization and under system optimization. That is,

$$
LOC = \frac{Z_{UO}}{Z_{SO}}. \tag{3.4}
$$

3.3 User Optimization

In this section, we will derive the equilibrium selling prices, wholesale prices, market demand and the total profits under user optimization.

We first relax the non-negativity constraints and solve the first order optimality conditions for the optimization problems. We then show that, under the assumptions we impose in Section 3.2.1, the solutions obtained satisfy the constraints and are thus feasible, and hence optimal, solutions to Optimization Problems (3.1) and (3.2).

Proposition 3.3.1 *Under Assumptions 3.2.1 and 3.2.5, the equilibrium total profit in user optimization is*

$$Z_{UO} = \frac{1}{4}(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c})^T(2\mathbf{B} + \mathbf{\Gamma})(\mathbf{B} + \mathbf{\Gamma})^{-1}\mathbf{B}(\mathbf{B} + \mathbf{\Gamma})^{-1}\mathbf{\Gamma}(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c}). \tag{3.5}$$

There exist unique equilibrium wholesale prices, $\mathbf{w_{UO}}$*, selling prices,* $\mathbf{p_{UO}}$*, and market demand,* $\mathbf{q_{UO}}$*, given respectively by*

$$\mathbf{w_{UO}} = \frac{1}{2}(\mathbf{B}^{-1}\mathbf{d} + \mathbf{c}). \tag{3.6}$$

$$\mathbf{p_{UO}} = \frac{1}{2}(\mathbf{B} + \mathbf{\Gamma})^{-1}(2\mathbf{B} + \mathbf{\Gamma})(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c}) + \mathbf{c}, \tag{3.7}$$

$$\mathbf{q_{UO}} = \frac{1}{2}\mathbf{B}(\mathbf{B} + \mathbf{\Gamma})^{-1}\mathbf{\Gamma}(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c}). \tag{3.8}$$

Proof Given the wholesale price vector w_i, imposed by the supplier, each retailer i decide on the selling price to maximize their own profit. Under an affine demand function, the market demand for retailer i is

$$q_i(p_i, \mathbf{p_{UO,-i}}) = d_i - \alpha_i p_i + \sum_{j \neq i} \beta_{i,j} p_j^{UO}.$$

With the above market demand, the profit for retailer i is

$$Z_{R_i} = (p_i - w_i)(d_i - \alpha_i p_i + \sum_{j \neq i} \beta_{i,j} p_j).$$

We relax the non-negativity constraints in Optimization Problem (3.1) to determine the best response pricing policy for retailer i, which is achieved when

$$\frac{\partial Z_{R_i}}{\partial p_i} = d_i - 2\alpha_i p_i + \sum_{j \neq i} \beta_{i,j} p_j + \alpha_i w_i = 0,$$

$$\nabla \mathbf{Z_R}(\mathbf{p}) = \mathbf{d} - \mathbf{Bp} - \mathbf{\Gamma p} + \mathbf{\Gamma w} = 0.$$

Therefore, the retailers' selling price is

$$\mathbf{p} = (\mathbf{B} + \mathbf{\Gamma})^{-1}(\mathbf{d} + \mathbf{\Gamma w}),$$

given the supplier's wholesale price vector $\mathbf{w}$. In particular, at Nash equilibrium, given the optimal wholesale price vector $\mathbf{w_{UO}}$, the equilibrium quantity is

$$\mathbf{p_{UO}} = (\mathbf{B} + \mathbf{\Gamma})^{-1}(\mathbf{d} + \mathbf{\Gamma w_{UO}}). \tag{3.9}$$

By Assumption 3.2.1, the equilibrium market demand, $\mathbf{q_{UO}}$, is

$$\mathbf{q_{UO}} = \mathbf{d} - \mathbf{B}(\mathbf{B} + \mathbf{\Gamma})^{-1}(\mathbf{d} + \mathbf{\Gamma w_{UO}}). \tag{3.10}$$

The supplier maximizes profit by deciding the wholesale prices as a best response to the equilibrium order quantities by the retailers. Given an equilibrium order quantity, $\mathbf{q_{UO}}$, the supplier makes a decision on the wholesale price to maximize his profit. The supplier's profit, $\mathbf{Z_S}$, is given by:

$$Z_S = (\mathbf{w} - \mathbf{c})^T \mathbf{q_{UO}} = (\mathbf{w} - \mathbf{c})^T [\mathbf{d} - \mathbf{B}(\mathbf{B} + \mathbf{\Gamma})^{-1}(\mathbf{d} + \mathbf{\Gamma w})].$$

We relax the non-negativity constraints in Optimization Problem (3.2) given the anticipated production quantities, to determine the supplier's optimal pricing policy. Optimality is achieved when

$$\nabla \mathbf{Z_S}(\mathbf{w}) = \mathbf{d} - \mathbf{B}(\mathbf{B} + \mathbf{\Gamma})^{-1}\mathbf{d} - \mathbf{B}(\mathbf{B} + \mathbf{\Gamma})^{-1}\mathbf{\Gamma w} - [\mathbf{B}(\mathbf{B} + \mathbf{\Gamma})^{-1}\mathbf{\Gamma}]^T(\mathbf{w} - \mathbf{c}) = 0.$$

Therefore, the supplier's optimal wholesale price is

$$\mathbf{w_{UO}} = [\mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma} + (\mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma})^T]^{-1}[\mathbf{d} - \mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{d} + (\mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma})^T\mathbf{c}]. \tag{3.11}$$

Under Assumption 3.2.5, which requires $\mathbf{B}$ to be a symmetric matrix, it follows that

$$\mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma} = [\mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma}]^T.$$

The supplier's optimal wholesale price therefore reduces to

$$\begin{aligned}
\mathbf{w_{UO}} &= [2\mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma}]^{-1}[\mathbf{d} - \mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{d} + \mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma}\mathbf{c}],\\
&= \frac{1}{2}[\mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma}]^{-1}\mathbf{d} - \frac{1}{2}\Gamma^{-1}\mathbf{d} + \frac{1}{2}\mathbf{c},\\
&= \frac{1}{2}[\mathbf{\Gamma}^{-1}(\mathbf{B}+\mathbf{\Gamma})\mathbf{B}^{-1} - \Gamma^{-1}]\mathbf{d} + \frac{1}{2}\mathbf{c},\\
&= \frac{1}{2}[\mathbf{\Gamma}^{-1}\mathbf{B}\mathbf{B}^{-1} + \mathbf{\Gamma}^{-1}\mathbf{\Gamma}\mathbf{B}^{-1} - \mathbf{\Gamma}^{-1}]\mathbf{d} + \frac{1}{2}\mathbf{c},\\
&= \frac{1}{2}(\mathbf{B}^{-1}\mathbf{d} + \mathbf{c}),\\
&\geq \mathbf{c} \quad \text{By Assumption 3.2.2.}
\end{aligned}$$

From the retailers' and supplier's optimal policies in Equation (3.10) and Equation (3.11), we derive the optimal quantities, prices and profits in terms of constant vectors, $\mathbf{d}$ and $\mathbf{c}$, and constant matrices, $\mathbf{B}$ and $\mathbf{\Gamma}$. Substituting Equation (3.6) into Equation (3.9), we obtain the equilibrium selling price under user optimization, given by the vector

$$\begin{aligned}
\mathbf{p_{UO}} &= (\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{d} + \frac{1}{2}\mathbf{\Gamma}(\mathbf{B}^{-1}\mathbf{d} + \mathbf{c})),\\
&= \frac{1}{2}(\mathbf{B}+\mathbf{\Gamma})^{-1}(2\mathbf{d} + \mathbf{\Gamma}(\mathbf{B}^{-1}\mathbf{d} + \mathbf{c})),
\end{aligned}$$

The equilibrium market demand vector $\mathbf{q_{UO}}$, can be obtained by directly substituting

Equation (3.7) into the price demand relationship in Assumption 3.2.1.

$$\begin{aligned}
\mathbf{q_{UO}} &= \mathbf{d} - \mathbf{B}\mathbf{p_{UO}}, \\
&= \mathbf{d} - \frac{1}{2}\mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}[2\mathbf{d} + \mathbf{\Gamma}(\mathbf{B}^{-1}\mathbf{d} + \mathbf{c})].
\end{aligned}$$

Let $\mathbf{u} = \mathbf{B}^{-1}\mathbf{d} - \mathbf{c}$, and consequently $\mathbf{d} = \mathbf{B}(\mathbf{u}+\mathbf{c})$. The equilibrium selling price can be expressed as

$$\begin{aligned}
\mathbf{p_{UO}} &= \frac{1}{2}(\mathbf{B}+\mathbf{\Gamma})^{-1}[2\mathbf{B}(\mathbf{u}+\mathbf{c}) + \mathbf{\Gamma}(\mathbf{u}+2\mathbf{c})], \\
&= \frac{1}{2}(\mathbf{B}+\mathbf{\Gamma})^{-1}[2(\mathbf{B}+\mathbf{\Gamma})\mathbf{c} + (2\mathbf{B}+\mathbf{\Gamma})\mathbf{u}], \\
&= \mathbf{c} + \frac{1}{2}(\mathbf{B}+\mathbf{\Gamma})^{-1}(2\mathbf{B}+\mathbf{\Gamma})\mathbf{u}, \\
&= \frac{1}{2}(\mathbf{B}+\mathbf{\Gamma})^{-1}(2\mathbf{B}+\mathbf{\Gamma})(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c}) + \mathbf{c}, \\
&\geq \mathbf{w_{UO}}.
\end{aligned}$$

Similarly, the equilibrium market demand can be expressed as

$$\begin{aligned}
\mathbf{q_{UO}} &= \mathbf{B}(\mathbf{u}+\mathbf{c}) - \mathbf{B}\mathbf{c} - \frac{1}{2}\mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}(2\mathbf{B}+\mathbf{\Gamma})\mathbf{u}, \\
&= \mathbf{B}\mathbf{u} - \frac{1}{2}\mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}(2\mathbf{B}+\mathbf{\Gamma})\mathbf{u}, \\
&= \mathbf{B}\mathbf{u} - \frac{1}{2}\mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}(2\mathbf{B}+2\mathbf{\Gamma})\mathbf{u} + \frac{1}{2}\mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma}\mathbf{u}, \\
&= \frac{1}{2}\mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma}\mathbf{u}, \\
&= \frac{1}{2}\mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma}(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c}), \\
&\geq 0 \quad \text{By Assumption 3.2.4.}
\end{aligned} \tag{3.12}$$

Therefore, the total profit generated in the market under user optimization, Z_{UO}, is

$$
\begin{aligned}
Z_{UO} &= (\mathbf{p_{UO}} - \mathbf{c})^T \mathbf{q_{UO}}, \\
&= [\mathbf{c} + \frac{1}{2}(\mathbf{B}+\mathbf{\Gamma})^{-1}(2\mathbf{B}+\mathbf{\Gamma})\mathbf{u} - \mathbf{c}]^T [\frac{1}{2}\mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma}\mathbf{u}], \\
&= \frac{1}{4}\mathbf{u}^T(2\mathbf{B}+\mathbf{\Gamma})(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma}\mathbf{u}, \\
&= \frac{1}{4}(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c})^T(2\mathbf{B}+\mathbf{\Gamma})(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{\Gamma}(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c}).
\end{aligned}
$$

∎

Remark We have shown that the solutions $\mathbf{q_{UO}}$ and $\mathbf{w_{UO}}$ satisfy the first order optimality conditions when there are no non-negativity constraints. Nevertheless, the non-negativity constraints due to Assumption 3.2.4. They are thus feasible solutions to the retailers' and supplier's optimization problems respectively.

3.4 System Optimization

We will derive the optimal prices, quantities and profits under system optimization. We first relax the non-negativity constraints and solve the first order optimality condition for the objective function. We then show that the solutions obtained satisfy the constraints and are thus feasible solutions to the optimization problem.

Proposition 3.4.1 *Under Assumptions 3.2.1 and 3.2.5, the optimal total profit generated by the system optimization is*

$$Z_{SO} = \frac{1}{4}(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c})^T\mathbf{B}(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c}). \tag{3.13}$$

There exist unique optimal order quantities, $\mathbf{q_{SO}}$, and selling prices, $\mathbf{p_{SO}}$, given respectively by

$$\mathbf{q_{SO}} = \frac{1}{2}\mathbf{B}(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c}), \tag{3.14}$$

$$\mathbf{p_{SO}} = \frac{1}{2}(\mathbf{B}^{-1}\mathbf{d} + \mathbf{c}). \tag{3.15}$$

Notice these are independent from the wholesale price.

Proof The total profit of the system is the sum of the retailers' profits and the supplier's profit. It is given by

$$\begin{aligned} Z &= (\mathbf{w} - \mathbf{c})^T \mathbf{q}(\mathbf{p}) + (\mathbf{p} - \mathbf{w})^T \mathbf{q}(\mathbf{p}), \\ &= (\mathbf{p} - \mathbf{c})^T \mathbf{q}(\mathbf{p}). \end{aligned}$$

The resulting optimization problem is

$$\begin{aligned} \textbf{SO:} \quad & \max_{\mathbf{p_{SO}}} \quad (\mathbf{p} - \mathbf{c})^T \mathbf{q}(\mathbf{p_{SO}}), \\ & \text{s.t.} \quad \mathbf{p_{SO}} \geq \mathbf{c}, \quad \mathbf{q}(\mathbf{p_{SO}}) \geq 0. \end{aligned} \tag{3.16}$$

Under an affine demand function, Z_{SO} is given by

$$Z_{SO} = (\mathbf{p_{SO}} - \mathbf{c})^T \mathbf{d} - \mathbf{B}\mathbf{p_{SO}}. \tag{3.17}$$

Optimality under no non-negativity constraint is achieved when

$$\begin{aligned} \nabla \mathbf{Z}(\mathbf{p_{SO}}) &= \mathbf{d} - (\mathbf{B} + \mathbf{B}^T)\mathbf{p_{SO}} + \mathbf{B}^T \mathbf{c} = 0, \\ \mathbf{p_{SO}} &= (\mathbf{B} + \mathbf{B}^T)^{-1}(\mathbf{d} + \mathbf{B}^T \mathbf{c}). \end{aligned} \tag{3.18}$$

Under Assumption 3.2.5, the optimal selling prices are given by the vector

$$\begin{aligned} \mathbf{p_{SO}} &= \frac{1}{2}\mathbf{B}^{-1}(\mathbf{d} + \mathbf{B}\mathbf{c}). \\ &= \frac{1}{2}(\mathbf{B}^{-1}\mathbf{d} + \mathbf{c}), \\ &\geq \mathbf{c} \quad \text{By Assumption 3.2.2.} \end{aligned}$$

Therefore, the equilibrium market demand is

$$
\begin{aligned}
\mathbf{q_{SO}} &= \mathbf{d} - \mathbf{B}[\frac{1}{2}(\mathbf{B}^{-1}\mathbf{d} + \mathbf{c})], \\
&= \frac{1}{2}(\mathbf{d} - \mathbf{B}\mathbf{c}), \\
&= \frac{1}{2}\mathbf{B}(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c}), \\
&\geq 0 \quad \text{By Assumption 3.2.2.}
\end{aligned}
$$

Substituting Equation (3.14) into Equation (3.17), we obtain the optimal total profit generated by the system optimization, given by

$$
\begin{aligned}
Z_{SO} &= [\frac{1}{2}\mathbf{B}^{-1}(\mathbf{d} + \mathbf{B}\mathbf{c}) - \mathbf{c}]^T[\frac{1}{2}(\mathbf{d} - \mathbf{B}\mathbf{c})], \\
&= [\frac{1}{2}\mathbf{B}^{-1}\mathbf{d} - \frac{1}{2}\mathbf{c}]^T[\frac{1}{2}(\mathbf{d} - \mathbf{B}\mathbf{c})], \\
&= \frac{1}{4}(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c})^T\mathbf{B}(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c}).
\end{aligned}
$$

∎

Remark

1. The wholesale price vector $\mathbf{w}$, is a internal transaction between the retailers and the supplier. Under system optimization, this transaction is executed within the system and therefore has no influence on system profits.

2. We have shown that the solution $\mathbf{q_{SO}}$ satisfies the first order optimality conditions, and the non-negativity constraint due to Assumption 3.2.2. Therefore, $\mathbf{q_{SO}}$ and $\mathbf{p_{SO}}$ are feasible solutions to the system optimization problem.

3.5 Loss of Coordination under a General Affine Demand Model

We first derive the exact loss of coordination in terms of the price sensitivity matrix. Subsequently we give a lower bound for the loss of coordination in terms of the maximum eigenvalue of the normalized price sensitivity matrix, $\mathbf{G} = \mathbf{B}^{\frac{1}{2}}(2\mathbf{B} + \mathbf{\Gamma})^{-1}\mathbf{B}^{\frac{1}{2}}$.

3.5.1 Loss of Coordination in terms of the Price Sensitivity Matrix

In the following lemma, we will express the loss of coordination in terms of a matrix $\mathbf{G}$ and vector $\mathbf{w}$.

Lemma 3.5.1 *Under Assumptions 3.2.1 and 3.2.5, the loss of coordination in a Bertrand competition with substitute products is given by:*

$$LOC = \frac{\mathbf{w}^T(\mathbf{I} - 2\mathbf{G})(\mathbf{G} - \mathbf{I})^{-2}\mathbf{w}}{\mathbf{w}^T\mathbf{w}}, \tag{3.19}$$

where

$$\mathbf{G} = \mathbf{B}^{\frac{1}{2}}(2\mathbf{B} + \mathbf{\Gamma})^{-1}\mathbf{B}^{\frac{1}{2}},$$

$$\mathbf{w} = (\mathbf{G} - \mathbf{I})\mathbf{B}^{\frac{1}{2}}(\mathbf{B} + \mathbf{\Gamma})^{-1}(2\mathbf{B} + \mathbf{\Gamma})(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c}).$$

Proof From Equation (3.5) and Equation (3.13),

$$LOC = \frac{(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c})^T(2\mathbf{B} + \mathbf{\Gamma})(\mathbf{B} + \mathbf{\Gamma})^{-1}\mathbf{B}(\mathbf{B} + \mathbf{\Gamma})^{-1}\mathbf{\Gamma}(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c})}{(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c})^T\mathbf{B}(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c})}. \tag{3.20}$$

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors such that

$$\mathbf{u} = (\mathbf{B}^{-1}\mathbf{d} - \mathbf{c}),$$

$$\mathbf{v} = \mathbf{B}^{\frac{1}{2}}(\mathbf{B} + \mathbf{\Gamma})^{-1}(2\mathbf{B} + \mathbf{\Gamma})(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c}),$$

and note that $\mathbf{v}^T = (\mathbf{B}^{-1}\mathbf{d} - \mathbf{c})^T(2\mathbf{B} + \mathbf{\Gamma})(\mathbf{B} + \mathbf{\Gamma})^{-1}\mathbf{B}^{\frac{1}{2}}$, since $\mathbf{B}$ and $\mathbf{\Gamma}$ are symmetric matrices. We can express $\mathbf{B}(\mathbf{B} + \mathbf{\Gamma})^{-1}\mathbf{\Gamma}$ as

$$\begin{aligned}\mathbf{B}(\mathbf{B} + \mathbf{\Gamma})^{-1}\mathbf{\Gamma} &= 2\mathbf{B} - 2\mathbf{B}(\mathbf{B} + \mathbf{\Gamma})^{-1}(\mathbf{B} + \mathbf{\Gamma}) + \mathbf{B}(\mathbf{B} + \mathbf{\Gamma})^{-1}\mathbf{\Gamma}, \\ &= 2\mathbf{B} - [\mathbf{B}(\mathbf{B} + \mathbf{\Gamma})^{-1}(2\mathbf{B} + 2\mathbf{\Gamma} - \mathbf{\Gamma})], \\ &= 2\mathbf{B} - \mathbf{B}(\mathbf{B} + \mathbf{\Gamma})^{-1}(2\mathbf{B} + \mathbf{\Gamma}).\end{aligned}$$

Therefore the equilibrium total profit under user optimization in Equation (3.5) is

$$
\begin{aligned}
Z_{UO} &= \frac{1}{4}\mathbf{u}^T(2\mathbf{B}+\mathbf{\Gamma})(\mathbf{B}+\mathbf{\Gamma})^{-1}[2\mathbf{B}-\mathbf{B}(\mathbf{B}+\mathbf{\Gamma})^{-1}(2\mathbf{B}+\mathbf{\Gamma})]\mathbf{u}, \\
&= \frac{1}{4}\mathbf{u}^T(2\mathbf{B}+\mathbf{\Gamma})(\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{B}^{\frac{1}{2}}[2\mathbf{B}^{\frac{1}{2}}-\mathbf{B}^{\frac{1}{2}}(\mathbf{B}+\mathbf{\Gamma})^{-1}(2\mathbf{B}+\mathbf{\Gamma})]\mathbf{u}, \\
&= \frac{1}{4}\mathbf{v}^T[2\mathbf{B}^{\frac{1}{2}}(2\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{B}+\mathbf{\Gamma})\mathbf{B}^{-\frac{1}{2}}-\mathbf{I}]\mathbf{B}^{\frac{1}{2}}(\mathbf{B}+\mathbf{\Gamma})^{-1}(2\mathbf{B}+\mathbf{\Gamma})\mathbf{u}, \\
&= \frac{1}{4}\mathbf{v}^T[2\mathbf{B}^{\frac{1}{2}}(2\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{B}+\mathbf{\Gamma})\mathbf{B}^{-\frac{1}{2}}-\mathbf{I}]\mathbf{v}, \\
&= \frac{1}{2}\mathbf{v}^T\mathbf{B}^{\frac{1}{2}}(2\mathbf{B}+\mathbf{\Gamma})^{-1}(2\mathbf{B}+\mathbf{\Gamma}-\mathbf{B})\mathbf{B}^{-\frac{1}{2}}\mathbf{v}-\frac{1}{4}\mathbf{v}^T\mathbf{v}, \\
&= \frac{1}{2}\mathbf{v}^T\mathbf{v}-\frac{1}{2}\mathbf{v}^T\mathbf{B}^{\frac{1}{2}}(2\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{B}\mathbf{B}^{-\frac{1}{2}}\mathbf{v}-\frac{1}{4}\mathbf{v}^T\mathbf{v}, \\
&= \frac{1}{4}\mathbf{v}^T[\mathbf{I}-2\mathbf{B}^{\frac{1}{2}}(2\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{B}^{\frac{1}{2}}]\mathbf{v}. \\
&= \frac{1}{4}\mathbf{v}^T(\mathbf{I}-2\mathbf{G})\mathbf{v}.
\end{aligned}
\tag{3.21}
$$

To express Z_{UO} in terms of $\mathbf{v}$ and $\mathbf{G}$, we need Equation (3.22) and Equation (3.23), as follows:

$$
\begin{aligned}
&\mathbf{B}^{-\frac{1}{2}}(-\mathbf{B})(2\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{B}(2\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{B}+\mathbf{\Gamma})\mathbf{B}^{-\frac{1}{2}} \\
=&\mathbf{B}^{\frac{1}{2}}(2\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{B}(2\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{B}-2\mathbf{B}-\mathbf{\Gamma})\mathbf{B}^{-\frac{1}{2}}, \\
=&\mathbf{B}^{\frac{1}{2}}(2\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{B}^{\frac{1}{2}}[\mathbf{B}^{\frac{1}{2}}(2\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{B}^{\frac{1}{2}}-\mathbf{B}^{\frac{1}{2}}(2\mathbf{B}+\mathbf{\Gamma})^{-1}(2\mathbf{B}+\mathbf{\Gamma})\mathbf{B}^{-\frac{1}{2}}], \\
=&\mathbf{G}(\mathbf{G}-\mathbf{I}).
\end{aligned}
\tag{3.22}
$$

Note that

$$
\begin{aligned}
&\mathbf{B}^{-\frac{1}{2}}(2\mathbf{B}+\mathbf{\Gamma})(2\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{B}(2\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{B}+\mathbf{\Gamma})\mathbf{B}^{-\frac{1}{2}} \\
=&\mathbf{B}^{\frac{1}{2}}(2\mathbf{B}+\mathbf{\Gamma})^{-1}(2\mathbf{B}+\mathbf{\Gamma}-\mathbf{B})\mathbf{B}^{-\frac{1}{2}}, \\
=&\mathbf{B}^{\frac{1}{2}}[\mathbf{I}-(2\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{B}]\mathbf{B}^{-\frac{1}{2}}, \\
=&\mathbf{I}-\mathbf{B}^{\frac{1}{2}}(2\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{B}^{\frac{1}{2}}, \\
=&\mathbf{I}-\mathbf{G}.
\end{aligned}
\tag{3.23}
$$

From Equation (3.22) and Equation (3.23), the total profit under system optimization is

$$
\begin{aligned}
Z_{SO} &= \frac{1}{4}\mathbf{u}^T\mathbf{B}\mathbf{u}, \\
&= \frac{1}{4}\mathbf{v}^T[\mathbf{B}^{-\frac{1}{2}}(\mathbf{B}+\boldsymbol{\Gamma})(2\mathbf{B}+\boldsymbol{\Gamma})^{-1}\mathbf{B}(2\mathbf{B}+\boldsymbol{\Gamma})^{-1}(\mathbf{B}+\boldsymbol{\Gamma})\mathbf{B}^{-\frac{1}{2}}]\mathbf{v}, \\
&= \frac{1}{4}\mathbf{v}^T[\mathbf{B}^{-\frac{1}{2}}(-\mathbf{B}+2\mathbf{B}+\boldsymbol{\Gamma})(2\mathbf{B}+\boldsymbol{\Gamma})^{-1}\mathbf{B}(2\mathbf{B}+\boldsymbol{\Gamma})^{-1}(\mathbf{B}+\boldsymbol{\Gamma})\mathbf{B}^{-\frac{1}{2}}]\mathbf{v}, \\
&= \frac{1}{4}\mathbf{v}^T\mathbf{G}(\mathbf{G}-\mathbf{I})\mathbf{v} + \frac{1}{4}\mathbf{v}^T(\mathbf{I}-\mathbf{G})\mathbf{v}, \\
&= \frac{1}{4}\mathbf{v}^T(\mathbf{G}-\mathbf{I})^2\mathbf{v}.
\end{aligned} \tag{3.24}
$$

Since

$$
\mathbf{w} = (\mathbf{G}-\mathbf{I})\mathbf{B}^{\frac{1}{2}}(\mathbf{B}+\boldsymbol{\Gamma})^{-1}(2\mathbf{B}+\boldsymbol{\Gamma})(\mathbf{B}^{-1}\mathbf{d}-\mathbf{c}) = (\mathbf{G}-\mathbf{I})\mathbf{v},
$$

we are now able to express the loss of coordination in terms of $\mathbf{w}$ and $\mathbf{G}$, illustrated as follows:

$$
\begin{aligned}
LOC &= \frac{\mathbf{v}^T(\mathbf{I}-2\mathbf{G})\mathbf{v}}{\mathbf{v}^T(\mathbf{G}-\mathbf{I})^2\mathbf{v}}, \\
&= \frac{\mathbf{w}^T(\mathbf{G}-\mathbf{I})^{-1}(\mathbf{I}-2\mathbf{G})(\mathbf{G}-\mathbf{I})^{-1}\mathbf{w}}{\mathbf{w}^T\mathbf{w}}. \\
&= \frac{\mathbf{w}^T[(\mathbf{G}-\mathbf{I})^{-1}-2(\mathbf{G}-\mathbf{I})^{-1}\mathbf{G}](\mathbf{G}-\mathbf{I})^{-1}\mathbf{w}}{\mathbf{w}^T\mathbf{w}}. \\
&= \frac{\mathbf{w}^T[(\mathbf{G}-\mathbf{I})^{-1}-2(\mathbf{I}-\mathbf{G}^{-1})^{-1}](\mathbf{G}-\mathbf{I})^{-1}\mathbf{w}}{\mathbf{w}^T\mathbf{w}}. \\
&= \frac{\mathbf{w}^T[(\mathbf{G}-\mathbf{I})^{-1}-2\mathbf{G}(\mathbf{G}-\mathbf{I})^{-1}](\mathbf{G}-\mathbf{I})^{-1}\mathbf{w}}{\mathbf{w}^T\mathbf{w}}. \\
&= \frac{\mathbf{w}^T(\mathbf{I}-2\mathbf{G})(\mathbf{G}-\mathbf{I})^{-2}\mathbf{w}}{\mathbf{w}^T\mathbf{w}}.
\end{aligned}
$$

∎

3.5.2 Upper and Lower Bounds for the Loss of Coordination

We first find an upper and lower bound in terms of the minimum and maximum eigenvalue of matrix $\mathbf{G}$ respectively.

Theorem 3.5.2 *Under Assumptions 3.2.1, 3.2.5, 3.2.6 and 3.2.7, the loss of coor-*

dination is bounded by

$$\frac{1-2\lambda_{max}(\mathbf{G})}{(\lambda_{max}(\mathbf{G})-1)^2} \leq LOC \leq \frac{1-2\lambda_{min}(\mathbf{G})}{(\lambda_{min}(\mathbf{G})-1)^2},$$

where $\mathbf{G} = \mathbf{B}^{\frac{1}{2}}(2\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{B}^{\frac{1}{2}}$.

Proof From Equation (3.19),

$$LOC = \frac{\mathbf{w}^T(\mathbf{I}-2\mathbf{G})(\mathbf{G}-\mathbf{I})^{-2}\mathbf{w}}{\mathbf{w}^T\mathbf{w}},$$

Since $(\mathbf{I}-2\mathbf{G})(\mathbf{G}-\mathbf{I})^{-2}$ is a symmetric matrix, it can be unitarily diagonalized as

$$(\mathbf{I}-2\mathbf{G})(\mathbf{G}-\mathbf{I})^{-2} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^T,$$

where $\mathbf{P}$ is a unitary matrix such that $\mathbf{P}^T\mathbf{P} = \mathbf{I}$, and $\mathbf{\Lambda}$ is a diagonal matrix consisting of the eigenvalues of $(\mathbf{I}-2\mathbf{G})(\mathbf{G}-\mathbf{I})^{-2}$. Let $\overline{\mathbf{w}} = \mathbf{P}^T\mathbf{w}$. Therefore,

$$\begin{aligned} LOC &= \frac{\mathbf{w}^T\mathbf{P}\mathbf{\Lambda}\mathbf{P}^T\mathbf{w}}{\mathbf{w}^T\mathbf{w}}, \\ &= \frac{\overline{\mathbf{w}}^T\mathbf{\Lambda}\overline{\mathbf{w}}}{\overline{\mathbf{w}}^T\overline{\mathbf{w}}}. \end{aligned}$$

Let $\lambda_i(\mathbf{G})$ denote the eigenvalues of matrix $\mathbf{G}$. For any vector $\overline{\mathbf{w}}$, a lower and upper bound for the numerator of the above expression is

$$\begin{aligned} \overline{\mathbf{w}}^T\mathbf{\Lambda}\overline{\mathbf{w}} &= \sum_{i=1}^{n}\lambda_i[(\mathbf{I}-2\mathbf{G})(\mathbf{G}-\mathbf{I})^{-2}]|\overline{w}_i|^2, \\ &\geq \sum_{i=1}^{n}\lambda_{\min}[(\mathbf{I}-2\mathbf{G})(\mathbf{G}-\mathbf{I})^{-2}]|\overline{w}_i|^2, \\ &= \lambda_{\min}[(\mathbf{I}-2\mathbf{G})(\mathbf{G}-\mathbf{I})^{-2}]\sum_{i=1}^{n}|\overline{w}_i|^2, \\ &= \lambda_{\min}[(\mathbf{I}-2\mathbf{G})(\mathbf{G}-\mathbf{I})^{-2}]\overline{\mathbf{w}}^T\overline{\mathbf{w}}. \end{aligned}$$

Similarly,

$$\overline{\mathbf{w}}^T \boldsymbol{\Lambda} \overline{\mathbf{w}} \leq \lambda_{\max}[(\mathbf{I} - 2\mathbf{G})(\mathbf{G} - \mathbf{I})^{-2}] \overline{\mathbf{w}}^T \overline{\mathbf{w}}.$$

It is clear that $\lambda[(\mathbf{I} - 2\mathbf{G})(\mathbf{G} - \mathbf{I})^{-2}] = \dfrac{1 - 2\lambda(\mathbf{G})}{(\lambda(\mathbf{G}) - 1)^2}$, which is decreasing for $0 \leq (\lambda(\mathbf{G}) \leq \frac{1}{2}$. Since we know that $0 \leq \lambda(\mathbf{G}) \leq \frac{1}{2}$, we have

$$\lambda_{\min}[(\mathbf{I} - 2\mathbf{G})(\mathbf{G} - \mathbf{I})^{-2}] = \frac{1 - 2\lambda_{\max}(\mathbf{G})}{(\lambda_{\max}(\mathbf{G}) - 1)^2}.$$

Similarly,

$$\lambda_{\max}[(\mathbf{I} - 2\mathbf{G})(\mathbf{G} - \mathbf{I})^{-2}] = \frac{1 - 2\lambda_{\min}(\mathbf{G})}{(\lambda_{\min}(\mathbf{G}) - 1)^2}.$$

The loss of coordination is therefore lower bounded by

$$\begin{aligned} LOC &\geq \lambda_{\min}[(\mathbf{I} - 2\mathbf{G})(\mathbf{G} - \mathbf{I})^{-2}] \\ &= \frac{1 - 2\lambda_{\max}(\mathbf{G})}{(\lambda_{\max}(\mathbf{G}) - 1)^2}. \end{aligned}$$

Similarly,

$$LOC \leq \frac{1 - 2\lambda_{\min}(\mathbf{G})}{(\lambda_{\min}(\mathbf{G}) - 1)^2}.$$

∎

We will now find a lower bound for the loss of coordination in terms of the price sensitivity ratio, which is easier to compute.

Theorem 3.5.3 *Under Assumptions 3.2.1, 3.2.5, 3.2.6, 3.2.7, the loss of coordination is lower bounded by*

$$LOC \geq \frac{3 + 2r_{max}(\mathbf{B})}{(2 + r_{max}(\mathbf{B}))^2}, \tag{3.25}$$

where $r_{max}(\mathbf{B})$ is the quantity sensitivity ratios for matrix $\mathbf{B}$, defined by

$$r_{max}(\mathbf{B}) = \max_i \sum_{j \neq i} \frac{|b_{i,j}|}{b_{i,i}}.$$

Proof Consider the matrix $\mathbf{\Gamma}^{-1}\mathbf{B}$, explicitly given by:

$$\mathbf{\Gamma}^{-1}\mathbf{B} = \begin{bmatrix} 1 & -\frac{\beta_{1,2}}{\alpha_1} & \cdots & \cdots & -\frac{\beta_{1,n}}{\alpha_1} \\ -\frac{\beta_{2,1}}{\alpha_2} & 1 & \cdots & \cdots & -\frac{\beta_{2,n}}{\alpha_2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -\frac{\beta_{n-1,1}}{\alpha_{n-1}} & \vdots & \ddots & 1 & -\frac{\beta_{n-1,n}}{\alpha_{n-1}} \\ -\frac{\beta_{n,1}}{\alpha_n} & \cdots & \cdots & -\frac{\beta_{n,n-1}}{\alpha_n} & 1 \end{bmatrix}.$$

Observe that $\mathbf{G}$ and $(2\mathbf{I}+\mathbf{B}^{-1}\mathbf{\Gamma})^{-1}$ are similar matrices, since

$$\mathbf{B}^{-\frac{1}{2}}\mathbf{G}\mathbf{B}^{\frac{1}{2}} = (2\mathbf{B}+\mathbf{\Gamma})^{-1}\mathbf{B} = (2\mathbf{I}+\mathbf{B}^{-1}\mathbf{\Gamma})^{-1}.$$

Hence, $(2\mathbf{I}+\mathbf{B}^{-1}\mathbf{\Gamma})^{-1}$ and $\mathbf{G}$ have the same eigenvalues. Therefore,

$$\begin{aligned} \lambda_{max}(\mathbf{G}) &= \frac{1}{2+\lambda_{min}(\mathbf{B}^{-1}\mathbf{\Gamma})}, \\ &= \frac{1}{2+\dfrac{1}{\lambda_{max}(\mathbf{\Gamma}^{-1}\mathbf{B})}}, \\ &= \frac{\lambda_{max}(\mathbf{\Gamma}^{-1}\mathbf{B})}{2\lambda_{max}(\mathbf{\Gamma}^{-1}\mathbf{B})+1}. \end{aligned}$$

Using Theorem 3.5.2,

$$\begin{aligned} LOC &\geq \frac{1-2\left(\dfrac{\lambda_{max}(\mathbf{\Gamma}^{-1}\mathbf{B})}{2\lambda_{max}(\mathbf{\Gamma}^{-1}\mathbf{B})+1}\right)}{\left(\dfrac{\lambda_{max}(\mathbf{\Gamma}^{-1}\mathbf{B})}{2\lambda_{max}(\mathbf{\Gamma}^{-1}\mathbf{B})+1}-1\right)^2}, \\ &= \frac{2\lambda_{max}(\mathbf{\Gamma}^{-1}\mathbf{B})+1}{(\lambda_{max}(\mathbf{\Gamma}^{-1}\mathbf{B})+1)^2}. \end{aligned}$$

By Gersgorin's Theorem, all eigenvalues of $\mathbf{\Gamma}^{-1}\mathbf{B}$ are located in at least one of the disks:

$$\{z : |z-(\mathbf{\Gamma}^{-1}\mathbf{B})_{i,i}|\} \leq \sum_{j\neq i}^{n} |(\mathbf{\Gamma}^{-1}\mathbf{B})_{i,j}|, \quad i=1,2,...,n.$$

Therefore, an upper bound for $\lambda_{max}(\mathbf{\Gamma}^{-1}\mathbf{B})$ is

$$\begin{aligned}
\lambda_{max}(\mathbf{\Gamma}^{-1}\mathbf{B}) &\leq 1 + \sum_{j \neq i}^{n} |(\mathbf{\Gamma}^{-1}\mathbf{B})_{i,j}|, \\
&= 1 + \sum_{j \neq i}^{n} \frac{\beta_{i,j}}{\alpha_i}, \\
&\leq 1 + \max_i \sum_{j \neq i}^{n} \frac{\beta_{i,j}}{\alpha_i}, \\
&= 1 + r_{max}(\mathbf{B})
\end{aligned}$$

We now have a lower bound for the loss of coordination in terms of the price sensitivity computed directly from $\mathbf{B}$, given by:

$$\begin{aligned}
LOC &\geq \frac{2(1 + r_{max}(\mathbf{B})) + 1}{(1 + r_{max}(\mathbf{B}) + 1)^2}, \\
&= \frac{3 + 2r_{max}(\mathbf{B})}{(2 + r_{max}(\mathbf{B}))^2}.
\end{aligned}$$

$\blacksquare$

Remark Let $LOC(r_{max}(\mathbf{B}))$ denote the lower bound derived in Theorem 3.5.3. Since $0 \leq r_{max}(\mathbf{B}) \leq 1$, the lower bound is bounded by $\frac{5}{9} \leq LOC(r_{max}(\mathbf{B})) \leq \frac{3}{4}$.

Discussion To examine the impact of an additional tier of supplier to an oligopoly market consisting of multiple retailers, we compare the lower bound derived in Theorem 3.5.3 with the lower bound for a single-tier Bertrand oligopoly market given by Sun (2006). The two lower bounds are shown in Figure 3-1. The lower bound in a single-tier Bertrand oligopoly market given by Sun (2006) is

$$LOC \geq \frac{4(1 - r_{max}(\mathbf{B}))}{(2 - r_{max}(\mathbf{B}))^2}.$$

The lower bound for LOC for a two-tier supply chain dominates the lower bound for LOC in a one-tier supply chain when r is small (e.g., when $rleq0.78$). However, as r is large (e.g., when $r \geq 0.78$), the lower bound for the LOC in a one-tier supply

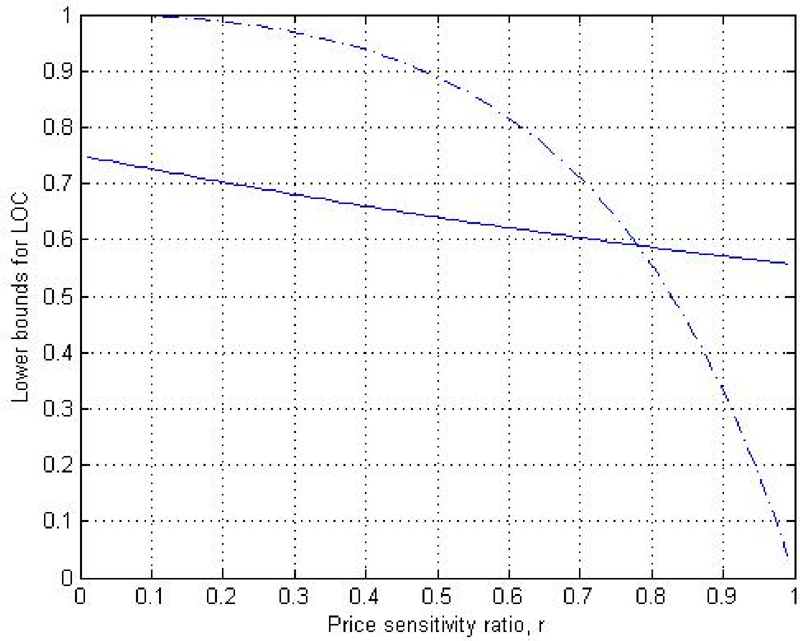

Figure 3-1: Comparing lower bounds in a one-tier and two-tier supply chain

chain dominates.

The value of r measures the intensity of competition. When r is small (e.g., when $r \leq 0.78$), competition is not intense. The presence of a supplier decreases the efficiency of the system (by incuring a greater loss of total profit) due to the lack of coordination between the supplier and the retailers. Under the extreme scenario when $r = 0$ (i.e. when there is no competition between retailers) the user optimization is equivalent to system optimization in a one-tier supply chain. However, in a two-tier supply chain, despite the absence of competition among retailers, there is lack of coordination between the supplier and retailer, since the supply chain in this case operates as if there were several independent supply chains of one supplier and one retailer. The loss of coordination, as a result, is given by the loss of coordination under the configuration of one supplier and one retailer, which is $\frac{3}{4}$.

When r is large (e.g., when $r \geq 0.78$), competition intensifies and the efficiency of the system deteoriates quickly in a one-tier supply chain. However, in a two-tier supply chain the presence of a supplier, who is the leader of the Stackelberg game, coordinates the supply chain under the same set of values of r. When r is large (e.g., when $r \geq 0.78$), the demand is inelastic (i.e., demand is not 'significantly' affected by prices), and the supplier is able to take advantage of the inelasticity of demand and set high wholesale prices, which has a significant impact on the pricing policies of the retailers. The pricing policy of the supplier is therefore coordinating the actions of the retailers, and hence coordinating the supply chain.

3.6 Tightness of Bounds

We analyze the tightness of the lower bounds in terms of the maximum eigenvalue of the quantity sensitivity matrix $\mathbf{B}$ by varying the number of retailers, n. We generate 10000 random instances of matrix $\mathbf{B}$, which satisfies the assumptions in Section 3.2.1, for each n (n varying from 2 to 20). We then obtain the averages of the loss of coordination and their bounds from these random instances. In these simulations, we consider two scenarios for vector $\mathbf{d}$:

1. $\mathbf{d} = \mathbf{c} + \mathbf{r}$, where $\mathbf{r}$ is a random vector of mean one.

2. $\mathbf{d} = \mathbf{c} + \mathbf{k}$, where $\mathbf{k}$ is a constant vector of ones.

Let the lower bound in terms of $\lambda_{min}(\mathbf{G})$ and $r_{max}(\mathbf{B})$ be denoted by $LOC(\lambda_{min}(\mathbf{G}))$ and $LOC(r_{max}(\mathbf{B}))$ respectively. The results of the simulations are shown in Figure 3-2 and 3-3.

Observations

1. The loss of total profit in the supply chain remains 'fairly' low for a varying number of retailers from 2 to 10. For this range of n considered, the average loss of total profit remains below 15%, while the maximum loss of total profit is consistently below 25%.

2. The lower bound remains 'fairly' tight. From the simulation, the lower bound differs from the actual average LOC by no more than 20%.

3. From the simulations, it seems that the average LOC is independent on the number of retailers, n.

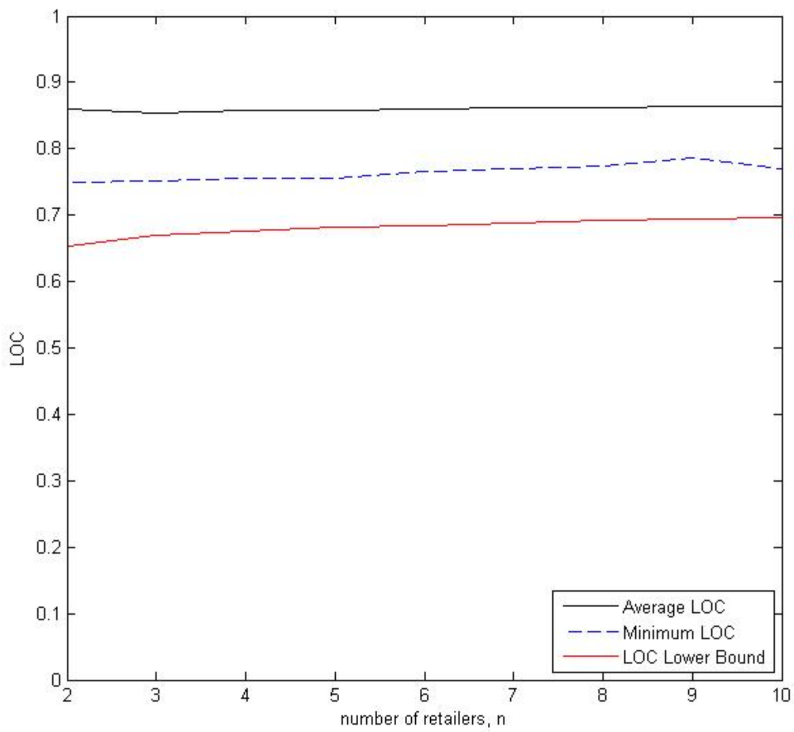

Figure 3-2: Lower Bounds with varying number of retailers, when $\mathbf{d} = \mathbf{Bc} + \mathbf{r}$, where $\mathbf{r}$ is a random vector of mean one.

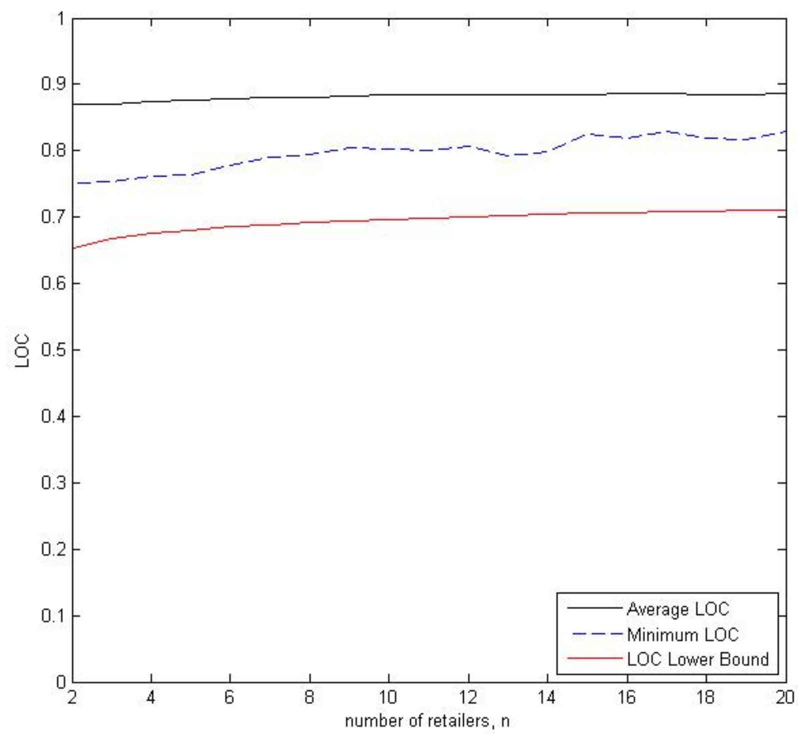

Figure 3-3: Lower Bounds with varying number of retailers, when $\mathbf{d} = \mathbf{Bc} + \mathbf{k}$, where $\mathbf{k}$ is a constant vector of ones.

4. The average LOC seems to be higher when $\mathbf{d} = \mathbf{c} + \mathbf{k}$, where $\mathbf{k}$ is a constant vector of ones. Moreover, the minimum LOC seems to be increasing with increasing n. However, these changes are small and barely noticeable.

3.7 Loss of Coordination under Uniform Demand

In this section, we analyze the loss of coordination in a symmetric setting under the uniform demand model without quality differences among products from different sellers.

3.7.1 Model Description

In this setting, all retailers encounter identical price sensitivities and the same demand function for all their products.

The following assumptions, in addition to those in Section 3.2.1, will be imposed throughout this section.

Assumption 3.7.1 *The price sensitivity is identical for all retailers. That is,* $\alpha_i = \alpha$, $\beta_i = \beta$, $q_i = d_i - \alpha p_i + \beta \mathbf{p_{-i}}$ *for all* $i = 1, 2, ..., n$. *Without loss of generality, we set* $\beta = 1$.

Assumption 3.7.2 *There is no quality differences between retailers. Moreover, the supplier incurs the same cost per unit quantity ordered by each retailer. That is,* $\mathbf{d} = (d, d, ..., d)^T$ *and* $\mathbf{c} = (c, c, ..., c)^T$.

Assumption 3.7.3 *The market clearing prices under zero production is at least as high as the per-unit costs incurred by the supplier, which must be non-negative. That is,* $\mathbf{d} \geq \mathbf{c} \geq 0$.

The vector $\mathbf{d}$ indicates the base demand prices (i.e., prices when quantities are zero) for the products by each retailer. If they are lower than the production costs, we can assume that the product is removed from the market in order for the firms to be profitable.

Recall that we are dealing with subsititute products in a price competition. Therefore, $\mathbf{B}$ is an M-matrix in the price-demand relationship $\mathbf{q} = \mathbf{d} - \mathbf{B}\mathbf{p}$. By Assumption 3.7.1, matrices $\mathbf{B}$ and $\mathbf{\Gamma}$ are

$$\mathbf{B} = \begin{bmatrix} \alpha & -1 & \cdots & \cdots & -1 \\ -1 & \alpha & \cdots & \cdots & -1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & \vdots & \ddots & \alpha & -1 \\ -1 & \cdots & \cdots & -1 & \alpha \end{bmatrix}, \quad \mathbf{\Gamma} = \begin{bmatrix} \alpha & 0 & \cdots & \cdots & 0 \\ 0 & \alpha & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \vdots & \ddots & \alpha & 0 \\ 0 & \cdots & \cdots & 0 & \alpha \end{bmatrix}.$$

3.7.2 User Optimization

We will present the equilibrium prices, quantities and profits under user optimization.

Proposition 3.7.4 *Under Assumption 3.7.1, 3.7.2 and 3.7.3, the equilibrium total profit generated under uniform demand in user optimization is*

$$Z_{UO} = \frac{\alpha(3\alpha + 2 - 2n)}{4(2\alpha + 1 - n)^2(\alpha + 1 - n)} n(d - (\alpha + 1 - n)c)^2, \tag{3.26}$$

with equilibrium wholesale prices, selling prices and market demand given respectively by the vectors

$$\mathbf{w_{UO}} = \frac{d + (\alpha + 1 - n)c}{2\alpha + 2 - 2n}\mathbf{e},$$

$$\mathbf{p_{UO}} = \frac{(3\alpha - 2n + 2)d + \alpha(\alpha + 1 - n)c}{(4\alpha + 2 - 2n)(\alpha + 1 - n)}\mathbf{e}, \tag{3.27}$$

$$\mathbf{q_{UO}} = \frac{\alpha d - \alpha(\alpha + 1 - n)c}{4\alpha + 2 - 2n}\mathbf{e}.$$

Proof First, note that we can express the matrices $\mathbf{B}$ and $\mathbf{B} + \mathbf{\Gamma}$ by

$$\mathbf{B} = (\alpha + 1)\mathbf{I} - \mathbf{H},$$

$$\mathbf{B} + \mathbf{\Gamma} = (2\alpha + 1)\mathbf{I} - \mathbf{H},$$

where

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & \cdots & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \vdots & \ddots & 1 & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{bmatrix}.$$

We rewrite matrices $\mathbf{B}$ and $(\mathbf{B}+\mathbf{\Gamma})^{-1}$ as follows:

$$\begin{aligned}\mathbf{B}^{-1} &= [(\alpha+1)\mathbf{I} - \mathbf{H}]^{-1}, \\ &= \frac{1}{\alpha+1}[\mathbf{I} - \frac{1}{\alpha+1}\mathbf{H}]^{-1}, \\ &= \frac{1}{\alpha+1}[\mathbf{I} + \frac{1}{\alpha+1}\mathbf{H} + (\frac{1}{\alpha+1}\mathbf{H})^2 + (\frac{1}{\alpha+1}\mathbf{H})^3 + ...].\end{aligned}$$

$$\begin{aligned}(\mathbf{B}+\mathbf{\Gamma})^{-1} &= [(2\alpha+1)\mathbf{I} - \mathbf{H}]^{-1}, \\ &= \frac{1}{2\alpha+1}[\mathbf{I} - \frac{1}{2\alpha+1}\mathbf{H}]^{-1}, \\ &= \frac{1}{2\alpha+1}[\mathbf{I} + \frac{1}{2\alpha+1}\mathbf{H} + (\frac{1}{2\alpha+1}\mathbf{H})^2 + (\frac{1}{2\alpha+1}\mathbf{H})^3 + ...].\end{aligned}$$

Since $\mathbf{H}^k = n^{k-1}\mathbf{H}$, it follows that

$$\begin{aligned}\mathbf{B}^{-1} &= \frac{1}{\alpha+1}[\mathbf{I} + \frac{1}{\alpha+1}\mathbf{H} + \frac{n}{(\alpha+1)^2}\mathbf{H} + \frac{n^2}{(\alpha+1)^3}\mathbf{H}...], \\ &= \frac{1}{\alpha+1}[\mathbf{I} + \frac{\frac{1}{\alpha+1}}{1-\frac{n}{\alpha+1}}\mathbf{H}], \\ &= \frac{1}{\alpha+1}[\mathbf{I} + \frac{1}{\alpha+1-n}\mathbf{H}].\end{aligned}$$

$$
\begin{aligned}
(\mathbf{B}+\mathbf{\Gamma})^{-1} &= \frac{1}{2\alpha+1}[\mathbf{I}+\frac{1}{2\alpha+1}\mathbf{H}+\frac{n}{(2\alpha+1)^2}\mathbf{H}+\frac{n^2}{(2\alpha+1)^3}\mathbf{H}...],\\
&= \frac{1}{2\alpha+1}[\mathbf{I}+\frac{\frac{1}{2\alpha+1}}{1-\frac{n}{2\alpha+1}}\mathbf{H}],\\
&= \frac{1}{2\alpha+1}[\mathbf{I}+\frac{1}{2\alpha+1-n}\mathbf{H}].
\end{aligned}
$$

From Equation (3.6),

$$
\begin{aligned}
\mathbf{w_{UO}} &= \frac{1}{2}(\mathbf{B}^{-1}\mathbf{d}+\mathbf{c}),\\
&= \frac{1}{2\alpha+2}(\mathbf{I}+\frac{1}{\alpha+1-n}\mathbf{H})\mathbf{d}+\frac{1}{2}\mathbf{c},\\
&= \left[\frac{1}{2\alpha+2}(1+\frac{n}{\alpha+1-n})d+\frac{1}{2}c\right]\mathbf{e},\\
&= \left(\frac{1}{2\alpha+2-2n}d+\frac{1}{2}c\right)\mathbf{e},\\
&= \frac{d+(\alpha+1-n)c}{2\alpha+2-2n}\mathbf{e}.
\end{aligned}
$$

From Equation (3.7),

$$
\begin{aligned}
\mathbf{p_{UO}} &= (\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{d}+\mathbf{\Gamma}\mathbf{w_{UO}}),\\
&= \frac{1}{2\alpha+1}[\mathbf{I}+\frac{1}{2\alpha+1-n}\mathbf{H}][\mathbf{d}+(\frac{1}{2\alpha+2-2n}d+\frac{1}{2}c)\mathbf{\Gamma}\mathbf{e}],\\
&= \frac{1}{2\alpha+1}[\mathbf{I}+\frac{1}{2\alpha+1-n}\mathbf{H}](d+\frac{\alpha}{2\alpha+2-2n}d+\frac{\alpha}{2}c)\mathbf{e},\\
&= \frac{1}{2\alpha+1}(1+\frac{n}{2\alpha+1-n})(d+\frac{\alpha}{2\alpha+2-2n}d+\frac{\alpha}{2}c)\mathbf{e},\\
&= \frac{1}{2\alpha+1-n}\left(\frac{3\alpha+2-2n}{2\alpha+2-2n}d+\frac{\alpha}{2}c\right)\mathbf{e},\\
&= \left(\frac{3\alpha-2n+2}{2(2\alpha+1-n)(\alpha+1-n)}d+\frac{\alpha}{2(2\alpha+1-n)}c\right)\mathbf{e},\\
&= \frac{(3\alpha-2n+2)d+\alpha(\alpha+1-n)c}{(4\alpha+2-2n)(\alpha+1-n)}\mathbf{e}.
\end{aligned}
$$

From the affine price demand relationship and Equation (3.27),

$$
\begin{aligned}
\mathbf{q_{UO}} &= \mathbf{d} - \mathbf{B}\mathbf{p_{UO}} \\
&= \mathbf{d} - [(\alpha+1)\mathbf{I} - \mathbf{H}]\left(\frac{3\alpha - 2n + 2}{2(2\alpha+1-n)(\alpha+1-n)}d + \frac{\alpha}{2(2\alpha+1-n)}c\right)\mathbf{e}, \\
&= \mathbf{d} - (\alpha+1-n)\left(\frac{3\alpha - 2n + 2}{2(2\alpha+1-n)(\alpha+1-n)}d + \frac{\alpha}{2(2\alpha+1-n)}c\right)\mathbf{e}, \\
&= \left(d - \frac{3\alpha - 2n + 2}{4\alpha + 2 - 2n}d - \frac{\alpha(\alpha+1-n)}{4\alpha+2-2n}c\right)\mathbf{e}, \\
&= \left(\frac{\alpha}{4\alpha+2-2n}d - \frac{\alpha(\alpha+1-n)}{4\alpha+2-2n}c\right)\mathbf{e}, \\
&= \frac{\alpha d - \alpha(\alpha+1-n)c}{4\alpha+2-2n}\mathbf{e}.
\end{aligned}
$$

The equilibrium total profit under user optimization is

$$
\begin{aligned}
Z_{UO} &= (\mathbf{p_{UO}} - \mathbf{c})^T\mathbf{q_{UO}}, \\
&= \frac{n(\alpha d - \alpha(\alpha+1-n)c)}{(4\alpha+2-2n)^2}\left(\frac{3\alpha-2n+2}{\alpha+1-n}d + \alpha c - (4\alpha+2-2n)c\right), \\
&= \frac{n((3\alpha-2n+2)d + (\alpha+1-n)(-3\alpha-2+2n)c)(\alpha d - \alpha(\alpha+1-n)c)}{(4\alpha+2-2n)^2(\alpha+1-n)}, \\
&= \frac{n(3\alpha-2n+2)(d-(\alpha+1-n)c)(\alpha d - \alpha(\alpha+1-n)c)}{(4\alpha+2-2n)^2(\alpha+1-n)}, \\
&= \frac{\alpha(3\alpha+2-2n)}{4(2\alpha+1-n)^2(\alpha+1-n)}n(d-(\alpha+1-n)c)^2.
\end{aligned}
$$

∎

3.7.3 System Optimization

We will present the optimal prices, quantities and profits under system optimization.

Proposition 3.7.5 *Under Assumption 3.7.1, 3.7.2 and 3.7.3, the equilibrium total profit generated under uniform demand in system optimization is*

$$Z_{SO} = \frac{n(d-(\alpha+1-n)c)^2}{4(\alpha+1-n)}. \tag{3.28}$$

with optimal selling prices and market demand given by the vectors

$$\mathbf{p_{SO}} = \frac{d + (\alpha + 1 - n)c}{2\alpha + 2 - 2n}\mathbf{e},$$

$$\mathbf{q_{SO}} = \frac{d - (\alpha + 1 - n)c}{2}\mathbf{e}.$$

Proof From Equation (3.15),

$$\begin{aligned}
\mathbf{p_{SO}} &= \frac{1}{2}(\mathbf{B}^{-1}\mathbf{d} + \mathbf{c}), \\
&= \frac{1}{2\alpha + 2}(\mathbf{I} + \frac{1}{\alpha + 1 - n}\mathbf{H})\mathbf{d} + \frac{1}{2}\mathbf{c}, \\
&= \left[\frac{1}{2\alpha + 2}(1 + \frac{n}{\alpha + 1 - n})d + \frac{1}{2}c\right]\mathbf{e}, \\
&= \left(\frac{1}{2\alpha + 2 - 2n}d + \frac{1}{2}c\right)\mathbf{e}, \\
&= \frac{d + (\alpha + 1 - n)c}{2\alpha + 2 - 2n}\mathbf{e}.
\end{aligned} \tag{3.29}$$

From the affine price demand relationship and Equation (3.29),

$$\begin{aligned}
\mathbf{q_{SO}} &= \mathbf{d} - \frac{d + (\alpha + 1 - n)c}{2\alpha + 2 - 2n}\mathbf{B}\mathbf{e}, \\
&= \mathbf{d} - \frac{d + (\alpha + 1 - n)c}{2\alpha + 2 - 2n}[(\alpha + 1)\mathbf{I} - \mathbf{H}]\mathbf{e}, \\
&= \mathbf{d} - \frac{d + (\alpha + 1 - n)c}{2\alpha + 2 - 2n}(\alpha + 1 - n)\mathbf{e}, \\
&= \left(d - \frac{d + (\alpha + 1 - n)c}{2}\right)\mathbf{e}, \\
&= \frac{d - (\alpha + 1 - n)c}{2}\mathbf{e}.
\end{aligned}$$

From Equation (3.13),

$$
\begin{aligned}
Z_{SO} &= \frac{1}{4}(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c})^T\mathbf{B}(\mathbf{B}^{-1}\mathbf{d} - \mathbf{c}),\\
&= \frac{1}{4}(\frac{d-(\alpha+1-n)c}{\alpha+1-n}\mathbf{e})^T\mathbf{B}(\frac{d-(\alpha+1-n)c}{\alpha+1-n}\mathbf{e}),\\
&= \frac{1}{4}(\frac{d-(\alpha+1-n)c}{\alpha+1-n}\mathbf{e})^T[(\alpha+1)\mathbf{I}-\mathbf{H}](\frac{d-(\alpha+1-n)c}{\alpha+1-n}\mathbf{e}),\\
&= \frac{n}{4}(\frac{d-(\alpha+1-n)c}{\alpha+1-n})(\alpha+1-n)(\frac{d-(\alpha+1-n)c}{\alpha+1-n}),\\
&= \frac{n(d-(\alpha+1-n)c)^2}{4(\alpha+1-n)}.
\end{aligned}
$$

$\blacksquare$

Remarks The optimal profit generated under system optimization is independent of the wholesale price, $\mathbf{w_{SO}}$.

3.7.4 Analysis of Loss of Coordination under Uniform Demand

We will study the efficency of the system under uniform demand by analyzing the loss of coordination. We will first give the expression for the loss of coordination in the following theorem:

Theorem 3.7.6 *Under Assumption 3.7.1, 3.7.2 and 3.7.3, the loss of coordination under a uniform demand function in a Bertrand competition with substitute products is*

$$LOC = \frac{3-2r}{(2-r)^2},$$

where $r = \frac{n-1}{\alpha}$.

Proof From Proposition 3.26 and Proposition 3.28, the loss of coordination is

$$
\begin{aligned}
LOC &= \frac{Z_{UO}}{Z_{SO}}, \\
&= \frac{\dfrac{\alpha(3\alpha+2-2n)n(d-(\alpha+1-n)c)^2}{4(2\alpha+1-n)^2(\alpha+1-n)}}{\dfrac{n(d-(\alpha+1-n)c)^2}{4(\alpha+1-n)}}, \\
&= \frac{\alpha(3\alpha+2-2n)}{(2\alpha+1-n)^2}, \\
&= \frac{(3+\dfrac{2-2n}{\alpha})}{(2+\dfrac{1-n}{\alpha})^2}, \\
&= \frac{3-2r}{(2-r)^2}.
\end{aligned}
$$

where $r = \dfrac{n-1}{\alpha}$.

∎

Remark Since $\lambda_{min}(\mathbf{G}) = 1 - r$ for uniform demand (i.e., symmetric retailers), the LOC upper bound in Theorem 3.5.2 is tight (i.e., it is achieved for symmetric retailers under uniform demand).

Discussions

Figure 3-4 shows the loss of coordination incurred by the supply chain under this model. There are two key observations regarding the loss of coordination in Theorem 3.7.6 as illustrated in Figure 3-4.

1. The supply chain is 'almost' coordinated when r is large. For example, $LOC \approx 99.2\%$ when $r = 0.9$. From the price demand relationship, the value of r indicates the inelasticity of demand, with the demand being perfectly elastic when $r = 0$, and perfectly inelastic (i.e., demand is independent of prices) when $r = 1$. As demand becomes 'more' inelastic, the supplier, being the leader, charges a very high wholesale price given by $\frac{1}{2}(\mathbf{p}(0) + \mathbf{c})$. Retailers, in response to high

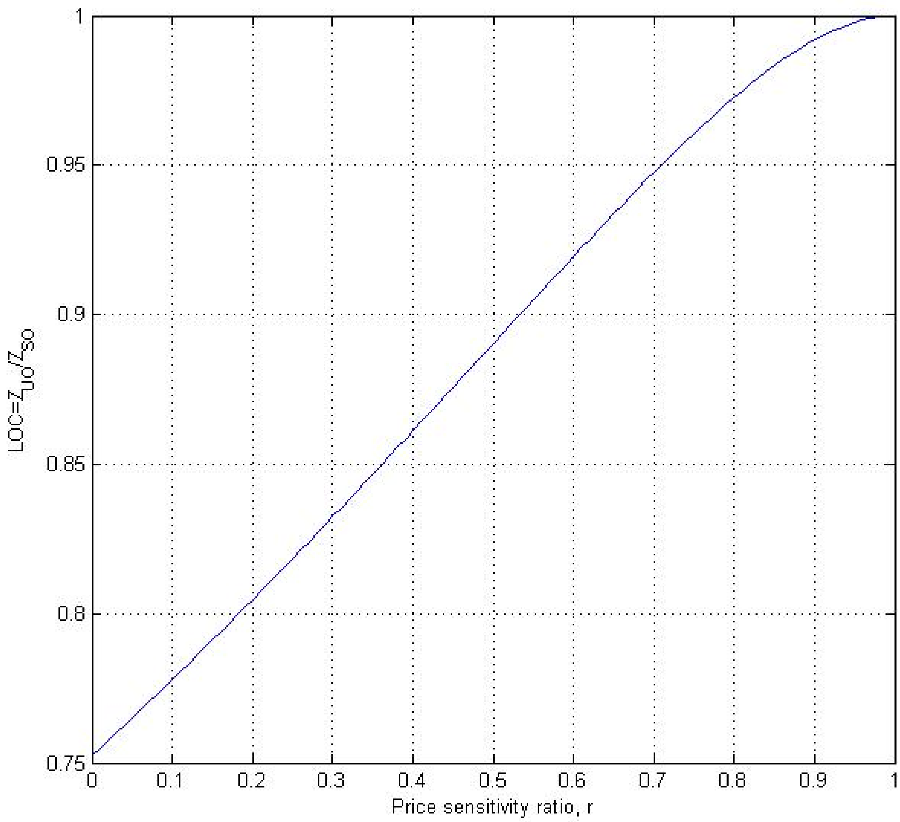

Figure 3-4: The loss of coordination in a Bertrand competition with substitute products and uniform demand.

wholesale prices, have to increase their selling prices. It can be shown that the total retailers' profit is finite under user optimization regardless of the elasticity of demand, and the ratio of the supplier's profit to the total retailers' profit is given by

$$\frac{Z_S^{UO}}{\sum_{i=1}^{n} Z_{R_i}^{UO}} = \frac{2-r}{1-r}.$$

This implies that the supplier receives at least 66% of the total supply chain profit, and this increases as r increases. An insight is that as demand becomes more inelastic, the supplier anticipates a small decrease in demand despite a significantly higher selling price. As the leader of the supply chain, he proposes a high wholesale price to the retailers, who in response, charges a high selling price to the consumers. The competiting retailers are unable to raise their selling price as high they wish, because competition will result in a lower market demand if their selling price is raised while competitors maintain their prices. As such, the supplier receives almost entirely the total supply chain profit, which approaches to that under system optimization.

2. The loss of coordination in Theorem 3.7.6 is strictly increasing with respect to r. By Assumption 3.2.6 and Assumption 3.2.7, we can deduce that $0 \leq \alpha \leq n-1$. Hence, $0 \leq r \leq 1$. Therefore, lower and upper bounds for the loss of coordination are attained when r takes values 0 and 1 respectively. We thus have

$$\frac{3}{4} \leq LOC \leq 1,$$

which is also illustrated by Figure 3-4.

3. The non-coordinated system is 'fairly' efficient. For instance, the loss of coordination stays within

$$0.89 \leq LOC \leq 1 \quad \text{for} \quad 0.5 \leq r \leq 1.$$

Chapter 4

Cournot Competition with Complement Products

4.1 Overview and Main Contributions

In this chapter, we analyze the loss of profit due to lack of coordination (we refer to this as loss of coordination) in a single-supplier, multi-retailer supply chain setting. The supply chain we consider is a Stackelberg game where the supplier is the leader and the retailers are the followers. The retailers compete in an oligopoly market through deciding quantities (Cournot competition) of complement products. Our model considers an affine demand price relation. This arises naturally from a quasilinear consumer utility function. As a special case, we also consider a uniform demand function, when all retailers encounter identical demand (i.e., have the same quantity sensitivities for all products). The demand function represents the consumers in an aggregate format and depends only on the quantities set by the retailers.

We evaluate the loss of coordination to measure the efficiency of the supply chain under competition, computed as the ratio of the total profit (that is, the total supplier's and retailers' profit) generated under competition (user optimization) and under coordination (system optimization). We also compare the policies under user optimization with that under system optimization. We then propose lower bounds for this loss of coordination to quantify the efficiency of the supply chain under competition. The lower bounds are in terms of the eigenvalues of the demand sensitivity matrix, or the demand sensitivities. One of these lower bounds indicate that the loss of coordination under uniform demand gives the worst case performance of the supply chain.

Theoretical and numerical simulations indicate that the supply chain is 'fairly' efficient for a large majority of the random data instances, even under uniform demand which gives the worst case performance. Further simulations under a general affine demand show that the actual performance of the supply chain is in fact much better than the worst case scenario under uniform demand, with average efficiency consistently above 50%.

The structure of the remainder of this chapter is as follows. Section 4.2 provides the groundwork for this chapter. Subsection 4.2.1 gives the notations and assumptions imposed in our analysis. We discuss the rationale and validity of these assumptions. In Subsection 4.2.2, we describe the model and review the central concepts of Nash equilibrium, user optimum, system optimum and the loss of coordination. In Section 4.3, we analyse the equilibrium and optimal prices, quantities and profits under user optimization and system optimization. In Subsection 4.3.1, we state the equilibrium wholesale prices, market clearing prices, order quantities and total profits under user optimization when individual market participants maximize their own profits, and state the optimal market clearing prices, order quantities and total profits achieved under system optimization when a central authority is coordinating decisions. In Subsection 4.3.2, we compare the prices and quantities obtained under user optimization against that under system optimization, and prove that the equlibrium quantities are non-negative. Section 4.4 presents the most important findings in this chapter - the loss of coordination in terms of the quantity sensitivity matrix, and presents lower bounds for this loss of coordination. We present three lower bounds, one in Subsection 4.4.1 which is in terms of the minimum eigenvalue of the quantity sensitivity matrix, and two lower bounds in Subsection 4.4.2 in terms of the quantity sensitivity ratios, which are easier to compute. In Section 4.5, we analyze the loss of coordination under the uniform demand model where all retailers encounter identical quantity sensitivities and experience the same demand function, and show that efficiency of the supply chain remains ‘fairly’ high for a large majority of randomly generated instances. Simulations are performed in Section 4.6 to evaluate and compare the tightness of these bounds. We show that the actual loss of efficiency of the supply chain is consistently below 48% on average, and performs much better than the supply chain under uniform demand. As an example, under the extreme scenario when competition between retailers are ‘very’ intense, the loss of efficiency for the uniform demand is 100%, while in a general affine demand, the loss of efficiency is about 48%.

4.2 Preliminaries

We consider a two tier single-supplier, multi-retailer supply chain producing differentiated complement products under an affine price demand relation competiting in a Cournot (quantity) oligopoly market, where retailers compete by deciding the quantities to produce and sell to the market at market clearing prices. We will first list the associated notations and assumptions in Section 4.2.1, and give more specific details of the model in Section 4.2.2.

4.2.1 Assumptions and Notations

In this supply chain with a single supplier and n retailers, we denote the order quantity of retailer i ($i = 1, 2, ..., n$) by q_i and let vector $\mathbf{q} = (q_1, ..., q_n)^T$. Similarly, let vectors $\mathbf{d}$, $\mathbf{c}$, $\mathbf{w}$ and $\mathbf{p}$ be the respective vectors for the market demand prices under zero production, the costs per unit order incurred by the supplier, the wholesale prices charged by the supplier and the market clearing prices. Let Z_{R_i} be the profit of retailer i, and Z_S be the supplier's profit.

Let the equilibrium wholesale prices, market clearing prices, production quantities and total profits under competition (user optimization) be denoted by $\mathbf{w_{UO}}$, $\mathbf{p_{UO}}$, $\mathbf{q_{UO}}$ and Z_{UO} respectively. Let the optimal market clearing prices, production quantities and total profits under coordination (system optimization) be denoted by $\mathbf{p_{SO}}$,

$\mathbf{q_{SO}}$ and Z_{SO} respectively.

Our analysis is restricted to models that satisfy the following assumptions:

Assumption 4.2.1 *The price demand relationship is affine and deterministic.*

This imply that the inverse demand function is $\mathbf{p}(\mathbf{q}) = \mathbf{d} - \mathbf{B}\mathbf{q}$, where $\mathbf{B}$ is the quantity sensitivity matrix. Affine demand functions are common in the pricing literature. Such a model arises naturally from a quasilinear utility function of a representative consumer. This model has been used by many researchers such as Carr et al. (1999), Berstein and Federgruen (2003), Allon and Federgruen (2006, 2007). In this thesis, we remove the effects of stochasticity of demand in order to isolate the effects of competition.

Assumption 4.2.2 *The market clearing prices under zero production is at least as high as the per-unit costs incurred by the supplier, which must be non-negative. That is,* $\mathbf{d} \geq \mathbf{c} \geq 0$.

The vector $\mathbf{d}$ indicates the base demand prices (i.e., prices when quantities are zero) for the products by each retailer. If they are lower than the production costs, we can assume that the product is removed from the market in order for the firms to be profitable.

Assumption 4.2.3 *The quantity sensitivity matrix,* $\mathbf{B}$*, is a symmetric matrix.*

This assumption implies that the cross-effects of the retailers' production quantities on each other are symmetric. This model arises naturally when a representative consumer maximizes a quasilinear utility function.

Assumption 4.2.4 *Matrix* $\mathbf{B}$ *has positive diagonals and non-negative off-diagonals.*

This is a natural consequence of a market with complement products. Increasing a retailer's production quantity has a negative effect on its own market prices, but a non-negative effect on other retailers' market prices.

Assumption 4.2.5 **B** *is a column-diagonally dominant matrix.*

This implies that a retailer's policy has a higher effect on its market prices than the total effect of the prices of all other retailers. This is applicable to markets where the sum of market prices decrease with an increase in quantity of one retailer.

Let **B** be the following matrix:

$$\mathbf{B} = \begin{bmatrix} \alpha_1 & -\beta_{1,2} & \cdots & \cdots & -\beta_{1,n} \\ -\beta_{2,1} & \alpha_2 & \cdots & \cdots & -\beta_{2,n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -\beta_{n-1,1} & \vdots & \ddots & \alpha_{n-1} & -\beta_{n-1,n} \\ -\beta_{n,1} & \cdots & \cdots & -\beta_{n,n-1} & \alpha_n \end{bmatrix},$$

and let $\mathbf{\Gamma}$ be a diagonal matrix consisting only of the diagonals of matrix **B**.

Remark Assumption 4.2.4 requires $\alpha_i > 0$ and $\beta_{i,j} \geq 0$ for all i, j. Assumption 4.2.5 requires $|\alpha_i| \geq \sum_{i \neq j} |\beta_{i,j}|$ for all i, j. These assumptions imply that **B** is an M-matrix, defined in Section 2.2.2.

For other definitions that will be used in this chapter, we refer the reader to Section 2.2.2.

4.2.2 Model Description

We consider a two tier single-supplier, multi-retailer supply chain producing differentiated complement products under an affine demand function. This is commonly employed in operations management literature (see for example, Carr et al. (1999), Berstein and Federgruen (2003), Allon and Federgruen (2006, 2007)).

The sequence of events is as follows. The supplier is a Stackelberg leader who first proposes a wholesale price to each of the retailers. After receiving the wholesale price, each retailer makes a decision on their own order quantities, and specifies to the supplier his/her respective order quantity. Upon receiving the order quantities, the supplier begins production and delivers items to each retailer at costs incurred by the supplier. The representative consumer will pay for all products available and therefore all quantities ordered by the retailers will be sold to the market.

In a Cournot (quantity) oligopoly market, retailer compete by deciding the quantity to produce and sell to the market at market clearing prices, which are determined as functions of the quantities sold through the inverse demand function (see Equation (2.1)).

Under user optimization, the supplier maximizes her profit by deciding the wholesale prices as a best response to the anticipated equilibrium order quantities by the retailers. The retailers decide on the quantities to sell to the market in response to the supplier's pricing policy. The supplier and each retailer is assumed to be rational and selfish, optimizing profits only for themselves. Nash Equilibrium is reached when no single retailer can increase its profit by unilaterally changing its production quantity.

For each retailer i, given the supplier's equilibrium wholesale price, w_i obtained from the vector $\mathbf{w_{UO}}$, and competitors' equilibrium quantities given by the vector $\mathbf{q_{UO,-i}}$, the retailer's best response quantity policy is obtained by solving the optimization problem $\mathbf{UO_{R_i}}$ described as follows:

$$\begin{aligned} \mathbf{UO_{R_i}}: \quad \max_{q_i} \quad & q_i.(p_i(q_i, \mathbf{q_{UO,-i}}) - w_i), \\ \text{s.t.} \quad & q_i \geq 0. \end{aligned} \tag{4.1}$$

The equilibrium wholesale price, w_i, for retailer i in the above problem is the solution to the supplier's optimization problem. The supplier maximizes revenue by deciding the wholesale prices vector $\mathbf{w_{UO}}$, given the retailers' equilibrium quantities obtained from vector $\mathbf{q_{UO}}$. This optimization problem, $\mathbf{UO_S}$, is described as follows:

$$\begin{aligned} \mathbf{UO_S}: \quad \max_{\mathbf{w}} \quad & \sum_{i=1}^{n} (w_i - c_i).q_i(w_i, \mathbf{w_{UO,-i}}), \\ \text{s.t.} \quad & q_i \geq 0, \quad \text{for all} \quad i = 1, ..., n. \end{aligned} \tag{4.2}$$

Let $Z_{R_i}^{UO}$ denote the profit of retailer i obtained from Optimization Problem (4.1), and Z_S^{UO} be the profit of the supplier obtained from Optimization Problem (4.2). The total profit under user optimization, Z_{UO}, is the sum of the profits of all the retailers and the supplier given by

$$Z_{UO} = Z_S^{UO} + \sum_{i=1}^{n} Z_{R_i}^{UO}.$$

Under system optimization, a central authority is coordinating all decisions, optimizing the total profit of the supplier and all retailers. The central authority makes decisions on all production quantities and forces the supplier and all retailers to comply. Coordination is attained by solving the following optimization problem, which determines the production quantities that maximize the total supply chain profit of the supplier and all retailers.

$$\begin{aligned} \textbf{SO:} \quad \max_{\mathbf{q_{SO}}} \quad & Z_S + \sum_{i=1}^{n} Z_{R_i}, \\ \text{s.t.} \quad & \mathbf{q_{SO}} \geq 0. \end{aligned} \tag{4.3}$$

Let Z_{SO} denote the optimal total profit obtained by solving the above optimization problem.

The loss of coordination, LOC, measures the loss of the total supply chain profit under competition, computed as the ratio of the total profit generated under user optimization and under system optimization. That is,

$$LOC = \frac{Z_{UO}}{Z_{SO}}. \tag{4.4}$$

4.3 Equilibrium and Optimal Quantities, Prices and Profits

With the equilibrium and optimal prices, quantities and profits established from Chapter 2, we prove the existence and uniqueness of these optimal solutions for both the user optimization and system optimization problems under our assumptions in this model. We then compare the prices and quantities obtained under user optimization and system optimization.

4.3.1 Solutions to Optimization Problems

The equilibrium and optimal prices, quantities and profits follow directly from Chapter 2 and are stated as follows:

Proposition 4.3.1 *Under Assumptions 4.2.1 and 4.2.3, the equilibrium total profit in user optimization is*

$$Z_{UO} = \frac{1}{4}(\mathbf{d}-\mathbf{c})^T[(\mathbf{B}+\boldsymbol{\Gamma})^{-1} + (\mathbf{B}+\boldsymbol{\Gamma})^{-1}\boldsymbol{\Gamma}(\mathbf{B}+\boldsymbol{\Gamma})^{-1}](\mathbf{d}-\mathbf{c}). \tag{4.5}$$

There exist unique equilibrium wholesale prices, $\mathbf{w_{UO}}$, production quantities, $\mathbf{q_{UO}}$, and market clearing prices, $\mathbf{p_{UO}}$, given respectively by

$$\mathbf{w_{UO}} = \frac{1}{2}(\mathbf{d}+\mathbf{c}). \tag{4.6}$$

$$\mathbf{q_{UO}} = \frac{1}{2}(\mathbf{B}+\boldsymbol{\Gamma})^{-1}(\mathbf{d}-\mathbf{c}), \tag{4.7}$$

$$\mathbf{p_{UO}} = \mathbf{d} - \frac{1}{2}\mathbf{B}(\mathbf{B}+\boldsymbol{\Gamma})^{-1}(\mathbf{d}-\mathbf{c}). \tag{4.8}$$

Proposition 4.3.2 *Under Assumptions 4.2.1 and 4.2.3, the optimal total profit generated by the system optimization is*

$$Z_{SO} = \frac{1}{4}(\mathbf{d}-\mathbf{c})^T\mathbf{B}^{-1}(\mathbf{d}-\mathbf{c}). \tag{4.9}$$

There exist unique optimal production quantities, $\mathbf{q_{SO}}$, and equilibrium market clearing prices, $\mathbf{p_{SO}}$, given respectively by

$$\mathbf{q_{SO}} = \frac{1}{2}\mathbf{B}^{-1}(\mathbf{d}-\mathbf{c}), \tag{4.10}$$

$$\mathbf{p_{SO}} = \frac{1}{2}(\mathbf{d}+\mathbf{c}), \tag{4.11}$$

and are independent of the wholesale price.

4.3.2 Price and Quantity Analysis

We will now prove in Lemma 4.3.3 that the constraints in the optimization problems arises naturally from the first order optimality conditions of the objective functions.

Lemma 4.3.3 *Under every assumption in Section 4.2.1, the equilibrium production quantities are non-negative. That is,*

$$\mathbf{q_{UO}} \geq 0.$$

Proof Note that $(\mathbf{B} + \mathbf{\Gamma})$ is an M-matrix since $\mathbf{B}$ is an M-matrix. Therefore, $\mathbf{c} \leq \mathbf{d}$ implies that

$$\begin{aligned} (\mathbf{B} + \mathbf{\Gamma})^{-T}\mathbf{c} &\leq (\mathbf{B} + \mathbf{\Gamma})^{-T}\mathbf{d}, \\ (\mathbf{B} + \mathbf{\Gamma})^{-1}\mathbf{d} + (\mathbf{B} + \mathbf{\Gamma})^{-T}\mathbf{c} &\leq (\mathbf{B} + \mathbf{\Gamma})^{-1}\mathbf{d} + (\mathbf{B} + \mathbf{\Gamma})^{-T}\mathbf{c} \qquad \leq \end{aligned} \tag{4.12}$$

As a result, the equilibrium wholesale price, $\mathbf{w_{UO}}$ is bounded above by the vector $\mathbf{d}$ as shown:

$$\mathbf{w_{UO}} = [(\mathbf{B} + \mathbf{\Gamma})^{-T} + (\mathbf{B} + \mathbf{\Gamma})^{-1}]^{-1}[(\mathbf{B} + \mathbf{\Gamma})^{-1}\mathbf{d} + (\mathbf{B} + \mathbf{\Gamma})^{-T}\mathbf{c}] \leq \mathbf{d}.$$

Since $\mathbf{d} - \mathbf{w_{UO}} \geq 0$,

$$\mathbf{q_{UO}} = (\mathbf{B} + \mathbf{\Gamma})^{-1}(\mathbf{d} - \mathbf{w_{UO}}) \geq 0.$$

∎

In the next proposition, we will compare the equilibrium market clearing prices and production quantities under user optimization and system optimization.

Proposition 4.3.4 *Under every assumption in Section 4.2.1, the market clearing prices under user optimization is at least as high as the prices under system optimization. That is,*

$$\mathbf{p_{UO}} \geq \mathbf{p_{SO}}.$$

Moreover, the production quantities under user optimization is at most equal to that under system optimization. That is,

$$\mathbf{q_{UO}} \leq \mathbf{q_{SO}}.$$

Proof Since $\mathbf{\Gamma} \geq 0$,

$$\mathbf{B} \leq \mathbf{B} + \mathbf{\Gamma}.$$

Since $(\mathbf{B} + \mathbf{\Gamma})$ is an M-matrix and $\mathbf{\Gamma} \geq 0$,

$$\mathbf{\Gamma}(\mathbf{B} + \mathbf{\Gamma})^{-1}(\mathbf{d} - \mathbf{c}) \geq 0,$$

$$\mathbf{d} - \mathbf{c} \geq \mathbf{d} - \mathbf{c} - \mathbf{\Gamma}(\mathbf{B} + \mathbf{\Gamma})^{-1}(\mathbf{d} - \mathbf{c}),$$

$$\mathbf{d} - \mathbf{c} \geq (\mathbf{B} + \mathbf{\Gamma})(\mathbf{B} + \mathbf{\Gamma})^{-1}\mathbf{d} - \mathbf{c} - \mathbf{\Gamma}(\mathbf{B} + \mathbf{\Gamma})^{-1}(\mathbf{d} - \mathbf{c}),$$

$$\mathbf{d} - \mathbf{c} \geq \mathbf{B}(\mathbf{B} + \mathbf{\Gamma})^{-1}(\mathbf{d} - \mathbf{c}),$$

$$-\frac{1}{2}(\mathbf{d} - \mathbf{c}) \leq -\frac{1}{2}\mathbf{B}(\mathbf{B} + \mathbf{\Gamma})^{-1}(\mathbf{d} - \mathbf{c}),$$

$$\mathbf{d} - \frac{1}{2}(\mathbf{d} - \mathbf{c}) \leq \mathbf{d} - \frac{1}{2}\mathbf{B}(\mathbf{B} + \mathbf{\Gamma})^{-1}(\mathbf{d} - \mathbf{c}),$$

$$\mathbf{p_{SO}} \leq \mathbf{p_{UO}}.$$

Since $\mathbf{B}$ is an M-matrix,

$$\mathbf{B}^T \leq \mathbf{\Gamma},$$

$$\mathbf{B} + \mathbf{B}^T \leq \mathbf{B} + \mathbf{\Gamma},$$

$$(\mathbf{B} + \mathbf{B}^T)^{-1} \geq (\mathbf{B} + \mathbf{\Gamma})^{-1}.$$

Observe that

$$\begin{aligned}
\mathbf{w_{UO}} &= \frac{1}{2}(\mathbf{d} + \mathbf{c}) \quad \text{by Equation (4.6)} \\
&\geq \frac{1}{2}(\mathbf{c} + \mathbf{c}) \quad \text{by Assumption 4.2.2} \\
&= \mathbf{c}
\end{aligned}$$

Therefore, we have $\mathbf{d} - \mathbf{w} \leq \mathbf{d} - \mathbf{c}$. It follows that

$$(\mathbf{B} + \mathbf{B}^T)^{-1}(\mathbf{d} - \mathbf{c}) \geq (\mathbf{B} + \mathbf{\Gamma})^{-1}(\mathbf{d} - \mathbf{w}),$$

$$\mathbf{q_{SO}} \geq \mathbf{q_{UO}}.$$

∎

4.4 Loss of Coordination in a General Affine Demand Model

We first state the loss of coordination in terms of the quantity sensitivity matrix from Chapter 2. We then give three lower bounds, one in terms of the minimum eigenvalue of the normalized quantity sensitivity matrix, and two in terms of the quantity sensitivity ratio.

From Chapter 2, we have the loss of coordination expressed as follows:

Theorem 4.4.1 *Under Assumptions 4.2.1 and 4.2.3, the loss of coordination in a supply chain with one supplier and n retailers under Cournot competition is*

$$LOC = \frac{(\mathbf{d} - \mathbf{c})^T[(\mathbf{B} + \mathbf{\Gamma})^{-1} + (\mathbf{B} + \mathbf{\Gamma})^{-1}\mathbf{\Gamma}(\mathbf{B} + \mathbf{\Gamma})^{-1}](\mathbf{d} - \mathbf{c})}{(\mathbf{d} - \mathbf{c})^T\mathbf{B}^{-1}(\mathbf{d} - \mathbf{c})}, \tag{4.13}$$

$$= \frac{\mathbf{w}^T(\mathbf{G} + 2\mathbf{I})\mathbf{w}}{\mathbf{w}^T(\mathbf{G} + \mathbf{G}^{-1} + 2\mathbf{I})\mathbf{w}}, \tag{4.14}$$

where $\mathbf{G} = \mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{B}\mathbf{\Gamma}^{-\frac{1}{2}}$ *and* $\mathbf{w} = \mathbf{\Gamma}^{\frac{1}{2}}(\mathbf{B} + \mathbf{\Gamma})^{-1}(\mathbf{d} - \mathbf{c})$.

4.4.1 Lower Bound in Terms of the Minimum Eigenvalue of G

We first find a lower bound in terms of the minimum eigenvalue of matrix $\mathbf{G}$. To do so, we will need the following proposition from Sun [1]:

Proposition 4.4.2 *Under Assumptions 4.2.4 and 4.2.5,*

$$\max_{\lambda_i(\mathbf{G})}\left[\lambda_i(\mathbf{G})+\frac{1}{\lambda_i(\mathbf{G})}+2\right]=\lambda_{min}(\mathbf{G})+\frac{1}{\lambda_{min}(\mathbf{G})}+2.$$

We are now ready to derive the lower bound, under the assumption that $\mathbf{B}$ is an M-matrix.

Theorem 4.4.3 *Under Assumptions 4.2.1, 4.2.3, 4.2.4 and 4.2.5,*

$$LOC \geq \frac{\lambda_{min}(\mathbf{G})(\lambda_{min}(\mathbf{G})+2)}{(\lambda_{min}(\mathbf{G})+1)^2}$$

Proof From Equation (4.14),

$$LOC=\frac{\mathbf{w}^T(\mathbf{G}+2\mathbf{I})\mathbf{w}}{\mathbf{w}^T(\mathbf{G}+\mathbf{G}^{-1}+2\mathbf{I})\mathbf{w}}.$$

Since $\mathbf{G}$ is a symmetric matrix, it can be unitarily diagonalized as:

$$\mathbf{G}=\mathbf{P}\mathbf{\Lambda}\mathbf{P}^T,$$

where $\mathbf{P}$ is a unitary matrix such that the Euclidean length of $\mathbf{P}^T\mathbf{w}$ is the same as that of $\mathbf{w}$, and $\mathbf{\Lambda}$ is a diagonal matrix consisting of the eigenvalues of $\mathbf{G}$. Therefore,

$$\begin{aligned}LOC &= \frac{\mathbf{w}^T(\mathbf{P}\mathbf{\Lambda}\mathbf{P}^T+2\mathbf{I})\mathbf{w}}{\mathbf{w}^T(\mathbf{P}\mathbf{\Lambda}\mathbf{P}^T+\mathbf{P}\mathbf{\Lambda}^{-1}\mathbf{P}^T+2\mathbf{I})\mathbf{w}},\\ &= \frac{\mathbf{w}^T(\mathbf{\Lambda}+2\mathbf{I})\mathbf{w}}{\mathbf{w}^T(\mathbf{\Lambda}+\mathbf{\Lambda}^{-1}+2\mathbf{I})\mathbf{w}}.\end{aligned}$$

Let $\lambda_i(\mathbf{G})$ denote the eigenvalues of $\mathbf{G}$. For any vector $\mathbf{w}$, a lower bound for the numerator of the above expression is

$$\begin{aligned}
\mathbf{w}^T(\mathbf{\Lambda}+2\mathbf{I})\mathbf{w} &= \sum_{i=1}^{n}\left[\lambda_i(\mathbf{G})+2\right]|w_i|^2, \\
&\geq \sum_{i=1}^{n}\left[\lambda_{min}(\mathbf{G})+2\right]|w_i|^2, \\
&= \left[\lambda_{min}(\mathbf{G})+2\right]\sum_{i=1}^{n}|w_i|^2, \\
&= [\lambda_{min}(\mathbf{G})+2]\mathbf{w}^T\mathbf{w}.
\end{aligned}$$

On the other hand, an upper bound for the denominator is

$$\begin{aligned}
\mathbf{w}^T(\mathbf{\Lambda}+\mathbf{\Lambda}^{-1}+2\mathbf{I})\mathbf{w} &= \sum_{i=1}^{n}\left[\lambda_i(\mathbf{G})+\frac{1}{\lambda_i(\mathbf{G})}+2\right]|w_i|^2, \\
&\leq \sum_{i=1}^{n}\max_{\lambda_i(\mathbf{G})}\left[\lambda_i(\mathbf{G})+\frac{1}{\lambda_i(\mathbf{G})}+2\right]|w_i|^2, \\
&= \sum_{i=1}^{n}[\lambda_{min}(\mathbf{G})+\frac{1}{\lambda_{min}(\mathbf{G})}+2]|w_i|^2, \quad \text{by Proposition 4.4.2} \\
&= [\lambda_{min}(\mathbf{G})+\frac{1}{\lambda_{min}(\mathbf{G})}+2]\mathbf{w}^T\mathbf{w}.
\end{aligned}$$

Therefore, a lower bound for the loss of coordination is

$$\begin{aligned}
LOC &\geq \frac{[\lambda_{min}(\mathbf{G})+2]\mathbf{w}^T\mathbf{w}}{[\lambda_{min}(\mathbf{G})+\frac{1}{\lambda_{min}(\mathbf{G})}+2]\mathbf{w}^T\mathbf{w}}, \\
&= \frac{\lambda_{min}(\mathbf{G})(\lambda_{min}(\mathbf{G})+2)}{(\lambda_{min}(\mathbf{G})+1)^2}.
\end{aligned}$$

∎

Remark The bound in Theorem 4.4.3 is tight when $\mathbf{w}=k\mathbf{v}$, where $\mathbf{v}$ is the eigenvector corresponding to $\lambda_{min}(\mathbf{G})$, and $k \in \Re$. This requires $\mathbf{d}=k(\mathbf{B}+\mathbf{\Gamma})\mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{v}+\mathbf{c}$.

4.4.2 Lower Bounds in Terms of Quantity Sensitivities

We will now find a lower bound for the loss of coordination in terms of the quantity sensitivity ratio, which is easier to compute.

Theorem 4.4.4 *Under Assumptions 4.2.1, 4.2.3, 4.2.4 and 4.2.5, the loss of coordination is lower bounded by*

$$LOC \geq \frac{(1 - r_{max}(\mathbf{G}))(3 - r_{max}(\mathbf{G}))}{(2 - r_{max}(\mathbf{G}))^2}, \tag{4.15}$$

$$\geq \frac{(1 - r_{max}(\mathbf{B}))(3 - r_{max}(\mathbf{B}))}{(2 - r_{max}(\mathbf{B}))^2}, \tag{4.16}$$

where $r_{max}(\mathbf{G})$ *and* $r_{max}(\mathbf{B})$ *are the quantity sensitivity ratios for matrices* $\mathbf{G}$ *and* $\mathbf{B}$ *defined by*

$$r_{max}(\mathbf{G}) = \max_i \sum_{j \neq i} |g_{i,j}|, \quad r_{max}(\mathbf{B}) = \max_i \sum_{j \neq i} \frac{|b_{i,j}|}{b_{i,i}}.$$

Proof By Gersgorin's Theorem (see Horn and Johnson (1985)), all eigenvalues of $\mathbf{G}$ are located in at least one of the disks:

$$\{z : |z - g_{i,i}|\} \leq \sum_{j \neq i}^{n} |g_{i,j}|, \quad i = 1, 2, ..., n.$$

Therefore, we find a lower bound for $\lambda_{min}(\mathbf{G})$ as follows:

$$\lambda_{min}(\mathbf{G}) - g_{i,i} \geq -\sum_{j \neq i}^{n} |g_{i,j}|, \quad i = 1, 2m, ..., n.$$

Since $g_{i,i} = 1$,

$$\lambda_{min}(\mathbf{G}) \geq 1 - \sum_{j \neq i}^{n} |g_{i,j}|,$$

$$\geq 1 - r_{max}(\mathbf{G}),$$

where $r_{max}(\mathbf{G}) = \max_i \sum_{j \neq i} |g_{i,j}|$. Furthermore, $\dfrac{\lambda_{min}(\mathbf{G})(\lambda_{min}(\mathbf{G}) + 2)}{(\lambda_{min}(\mathbf{G}) + 1)^2}$ is increasing in $\lambda_{min}(\mathbf{G})$ for $\lambda_{min}(\mathbf{G}) \geq 0$. Therefore,

$$\begin{aligned}
LOC &\geq \frac{\lambda_{min}(\mathbf{G})(\lambda_{min}(\mathbf{G})+2)}{(\lambda_{min}(\mathbf{G})+1)^2}, \\
&\geq \frac{(1-r_{max}(\mathbf{G}))(1-r_{max}(\mathbf{G})+2)}{(1-r_{max}(\mathbf{G})+1)^2}, \\
&= \frac{(1-r_{max}(\mathbf{G}))(3-r_{max}(\mathbf{G}))}{(2-r_{max}(\mathbf{G}))^2}.
\end{aligned}$$

Next, consider the matrix $\mathbf{\Gamma}^{-1}\mathbf{B}$, explicitly given by:

$$\mathbf{\Gamma}^{-1}\mathbf{B} = \begin{bmatrix}
1 & -\frac{\beta_{1,2}}{\alpha_1} & \cdots & \cdots & -\frac{\beta_{1,n}}{\alpha_1} \\
-\frac{\beta_{2,1}}{\alpha_2} & 1 & \cdots & \cdots & -\frac{\beta_{2,n}}{\alpha_2} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-\frac{\beta_{n-1,1}}{\alpha_{n-1}} & \vdots & \ddots & 1 & -\frac{\beta_{n-1,n}}{\alpha_{n-1}} \\
-\frac{\beta_{n,1}}{\alpha_n} & \cdots & \cdots & -\frac{\beta_{n,n-1}}{\alpha_n} & 1
\end{bmatrix}.$$

Observe that $\mathbf{\Gamma}^{-1}\mathbf{B}$ and $\mathbf{G}$ are similar matrices, since

$$\mathbf{G} = \mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{B}\mathbf{\Gamma}^{-\frac{1}{2}} = \mathbf{\Gamma}^{\frac{1}{2}}[\mathbf{\Gamma}^{-1}\mathbf{B}]\mathbf{\Gamma}^{-\frac{1}{2}}.$$

Hence, $\mathbf{\Gamma}^{-1}\mathbf{B}$ and $\mathbf{G}$ have the same eigenvalues. Using Theorem 4.4.3,

$$\begin{aligned}
LOC &\geq \frac{\lambda_{min}(\mathbf{G})(\lambda_{min}(\mathbf{G})+2)}{(\lambda_{min}(\mathbf{G})+1)^2}, \\
&= \frac{\lambda_{min}(\mathbf{\Gamma}^{-1}\mathbf{B})(\lambda_{min}(\mathbf{\Gamma}^{-1}\mathbf{B})+2)}{(\lambda_{min}(\mathbf{\Gamma}^{-1}\mathbf{B})+1)^2}.
\end{aligned}$$

By Gersgorin's Theorem, all eigenvalues of $\mathbf{\Gamma}^{-1}\mathbf{B}$ are located in at least one of the disks:

$$\{z : |z - (\mathbf{\Gamma}^{-1}\mathbf{B})_{i,i}|\} \leq \sum_{j \neq i}^{n} |(\mathbf{\Gamma}^{-1}\mathbf{B})_{i,j}|, \quad i = 1, 2, ..., n.$$

Therefore, a lower bound for $\lambda_{min}(\mathbf{\Gamma}^{-1}\mathbf{B})$ is

$$\begin{aligned}
\lambda_{min}(\mathbf{\Gamma}^{-1}\mathbf{B}) &\geq 1 - \sum_{j \neq i}^{n} |(\mathbf{\Gamma}^{-1}\mathbf{B})_{i,j}|, \\
&= 1 - \sum_{j \neq i}^{n} \frac{\beta_{i,j}}{\alpha_i}, \\
&\geq 1 - \max_i \sum_{j \neq i}^{n} \frac{\beta_{i,j}}{\alpha_i}, \\
&= 1 - r_{max}(\mathbf{B})
\end{aligned}$$

With the above inequality, we have a lower bound for the loss of coordination in terms of the quantity sensitivity computed directly from $\mathbf{B}$, given by:

$$\begin{aligned}
LOC &\geq \frac{(1 - r_{max}(\mathbf{B}))(1 - r_{max}(\mathbf{B}) + 2)}{(1 - r_{max}(\mathbf{B}) + 1)^2}, \\
&= \frac{(1 - r_{max}(\mathbf{B}))(3 - r_{max}(\mathbf{B}))}{(2 - r_{max}(\mathbf{B}))^2}.
\end{aligned}$$

∎

Discussion

1. The supply chain is less efficient when retailers are under intense competition, measured by the values of r. When r is large, the competitors' quantity policies has a greater effect on a retailer's market clearing price, suggesting a greater intensity of competition. Therefore, a 'large' r_{max} indicates 'intense' competition, which drives low the loss of coordination. On the other hand, when there is 'little' or no competition, the supply chain remains 'fairly' efficient, indicating high profits despite a lack of coordination.

4.5 Loss of Coordination under Uniform Demand

In this section, we analyze the loss of coordination in the symmetric setting under the uniform demand model without quality differences among products from different retailers. We show that the lower bound given in Inequality (4.16) is achieved under the uniform demand model, and therefore gives the worst case scenario for the loss of coordination.

4.5.1 Model Description

In this setting, all retailers encounter identical quantity sensitivities and the same demand function for all their products which has no quality differences.
The following assumptions, in addition to those in Section 4.2.1, will be imposed throughout this section.

Assumption 4.5.1 *The quantity sensitivity is identical for all retailers. That is,* $\alpha_i = \alpha$, $\beta_i = \beta$, $p_i = d_i - \alpha q_i + \beta \mathbf{q_{-i}}$ *for all* $i = 1, 2, ..., n$. *Without loss of generality, we set* $\beta = 1$.

Assumption 4.5.2 *There is no quality differences between retailers. Moreover, the supplier incurs the same cost per unit quantity ordered by each retailer. That is,* $\mathbf{d} = (d, d, ..., d)^T$ *and* $\mathbf{c} = (c, c, ..., c)^T$.

Assumption 4.5.3 *The market clearing prices under zero production is at least as high as the per-unit costs incurred by the supplier, which must be non-negative. That is,* $\mathbf{d} \geq \mathbf{c} \geq 0$.

The vector $\mathbf{d}$ indicates the base demand prices (i.e., prices when quantities are zero) for the products by each retailer. If they are lower than the production costs, we can assume that the product is removed from the market in order for the firms to be profitable.

Recall that we are dealing with complement products in a quantity competition. Therefore, $\mathbf{B}$ is an M-matrix in the price-demand relationship, where $\mathbf{p} = \mathbf{d} - \mathbf{Bq}$. By Assumption 4.5.1, matrices $\mathbf{B}$ and $\mathbf{\Gamma}$ are

$$\mathbf{B} = \begin{bmatrix} \alpha & -1 & \cdots & \cdots & -1 \\ -1 & \alpha & \cdots & \cdots & -1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & \vdots & \ddots & \alpha & -1 \\ -1 & \cdots & \cdots & -1 & \alpha \end{bmatrix}, \quad \mathbf{\Gamma} = \begin{bmatrix} \alpha & 0 & \cdots & \cdots & 0 \\ 0 & \alpha & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \vdots & \ddots & \alpha & 0 \\ 0 & \cdots & \cdots & 0 & \alpha \end{bmatrix}.$$

4.5.2 User Optimization

We will present the equilibrium prices, quantities and profits under user optimization.

Proposition 4.5.4 *Under Assumption 4.5.1, 4.5.2 and 4.5.3, the equilibrium total profit generated under uniform demand in user optimization is*

$$Z_{UO} = \frac{3\alpha + 1 - n}{4(2\alpha + 1 - n)^2} n(d - c)^2, \tag{4.17}$$

with equilibrium quantities, market clearing prices and wholesale prices respectively given by the vectors:

$$\mathbf{q_{UO}} = \frac{d - c}{4\alpha + 2 - 2n}\mathbf{e}, \tag{4.18}$$

$$\mathbf{p_{UO}} = \frac{(3\alpha + 1 - n)d + (\alpha + 1 - n)c}{4\alpha + 2 - 2n}\mathbf{e}, \tag{4.19}$$

$$\mathbf{w_{UO}} = \frac{d + c}{2}\mathbf{e}.$$

Proof First, observe that $\mathbf{w_{UO}} = \dfrac{d + c}{2}\mathbf{e}$ follows directly from Equation (4.6). Also note that we can express the matrices $\mathbf{B}$ and $\mathbf{B} + \mathbf{\Gamma}$ by

$$\mathbf{B} = (\alpha + 1)\mathbf{I} - \mathbf{H},$$

$$\mathbf{B} + \mathbf{\Gamma} = (2\alpha + 1)\mathbf{I} - \mathbf{H},$$

where

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & \cdots & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \vdots & \ddots & 1 & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{bmatrix}.$$

We rewrite matrix $(\mathbf{B}+\mathbf{\Gamma})^{-1}$ as follows:

$$\begin{aligned}
(\mathbf{B}+\mathbf{\Gamma})^{-1} &= [(2\alpha+1)\mathbf{I}-\mathbf{H}]^{-1}, \\
&= \frac{1}{2\alpha+1}[\mathbf{I}-\frac{1}{2\alpha+1}\mathbf{H}]^{-1}, \\
&= \frac{1}{2\alpha+1}[\mathbf{I}+\frac{1}{2\alpha+1}\mathbf{H}+(\frac{1}{2\alpha+1}\mathbf{H})^2+(\frac{1}{2\alpha+1}\mathbf{H})^3+...].
\end{aligned}$$

Since $\mathbf{H}^k = n^{k-1}\mathbf{H}$, it follows that

$$\begin{aligned}
(\mathbf{B}+\mathbf{\Gamma})^{-1} &= \frac{1}{2\alpha+1}[\mathbf{I}+\frac{1}{2\alpha+1}\mathbf{H}+\frac{n}{(2\alpha+1)^2}\mathbf{H}+\frac{n^2}{(2\alpha+1)^3}\mathbf{H}...], \\
&= \frac{1}{2\alpha+1}[\mathbf{I}+\frac{\frac{1}{2\alpha+1}}{1-\frac{n}{2\alpha+1}}\mathbf{H}], \\
&= \frac{1}{2\alpha+1}[\mathbf{I}+\frac{1}{2\alpha+1-n}\mathbf{H}].
\end{aligned}$$

From Equation (4.7),

$$\begin{aligned}
\mathbf{q_{UO}} &= \frac{1}{2}(\mathbf{B}+\mathbf{\Gamma})^{-1}(\mathbf{d}-\mathbf{c}), \\
&= \frac{1}{2}\frac{1}{2\alpha+1}[\mathbf{I}+\frac{1}{2\alpha+1-n}\mathbf{H}](\mathbf{d}-\mathbf{c}), \\
&= \frac{1}{4\alpha+2}(1+\frac{n}{2\alpha+1-n})(d-c)\mathbf{e}, \\
&= \frac{d-c}{4\alpha+2-2n}\mathbf{e}.
\end{aligned}$$

The equilibrium market clearing prices follows directly from the equilibrium quantities and price demand relationship assumed in Assumption 4.2.1. From the affine price demand relationship and Equation (4.18),

$$\begin{aligned}
\mathbf{p_{UO}} &= \mathbf{d} - \mathbf{B}\mathbf{q_{UO}} \\
&= \mathbf{d} - [(\alpha+1)\mathbf{I} - \mathbf{H}](\frac{d-c}{4\alpha+2-2n})\mathbf{e} \\
&= \left[d - (\alpha+1-n)\frac{d-c}{4\alpha+2-2n} \right] \mathbf{e} \\
&= \frac{(3\alpha+1-n)d + (\alpha+1-n)c}{4\alpha+2-2n}\mathbf{e}.
\end{aligned}$$

Therefore, the equilibrium total profit under user optimization is

$$\begin{aligned}
Z_{UO} &= (\mathbf{p_{UO}} - \mathbf{c})^T \mathbf{q_{UO}}, \\
&= n \left(\frac{(3\alpha+1-n)d + (\alpha+1-n)c}{4\alpha+2-2n} - c \right) \frac{d-c}{4\alpha+2-2n}, \\
&= \frac{n(d-c)}{(4\alpha+2-2n)^2}(3\alpha+1-n)(d-c), \\
&= \frac{3\alpha+1-n}{4(2\alpha+1-n)^2} n(d-c)^2.
\end{aligned}$$

$\blacksquare$

4.5.3 System Optimization

We will present the optimal prices, quantities and profits under system optimization.

Proposition 4.5.5 *Under Assumption 4.5.1, 4.5.2 and 4.5.3, the equilibrium total profit generated under uniform demand in system optimization is*

$$Z_{SO} = \frac{n(d-c)^2}{4\alpha+4-4n}, \tag{4.20}$$

with optimal production quantities and market clearing prices given by the vectors

$$\mathbf{q_{SO}} = \frac{d-c}{2\alpha+2-2n}\mathbf{e}. \tag{4.21}$$

$$\mathbf{p_{SO}} = \frac{d+c}{2}\mathbf{e}.$$

Proof We adopt a similar technique as in the proof of Theorem 4.18, and express $\mathbf{B}^{-1}$ as follows:

$$
\begin{aligned}
\mathbf{B}^{-1} &= [(\alpha+1)\mathbf{I} - \mathbf{H}]^{-1}, \\
&= \frac{1}{\alpha+1}[\mathbf{I} - \frac{1}{\alpha+1}\mathbf{H}]^{-1}, \\
&= \frac{1}{\alpha+1}[\mathbf{I} + \frac{1}{\alpha+1}\mathbf{H} + (\frac{1}{\alpha+1}\mathbf{H})^2 + (\frac{1}{\alpha+1}\mathbf{H})^3 + ...]. \\
&= \frac{1}{\alpha+1}[\mathbf{I} + \frac{1}{\alpha+1}\mathbf{H} + \frac{n}{(\alpha+1)^2}\mathbf{H} + \frac{n^2}{(\alpha+1)^3}\mathbf{H}...], \\
&= \frac{1}{\alpha+1}[\mathbf{I} + \frac{\frac{1}{\alpha+1}}{1-\frac{n}{\alpha+1}}\mathbf{H}], \\
&= \frac{1}{\alpha+1}[\mathbf{I} + \frac{1}{\alpha+1-n}\mathbf{H}].
\end{aligned}
$$

From Equation (4.10),

$$
\begin{aligned}
\mathbf{q_{SO}} &= \frac{1}{2}\mathbf{B}^{-1}(\mathbf{d}-\mathbf{c}), \\
&= \frac{1}{2}\frac{1}{\alpha+1}[\mathbf{I} + \frac{1}{\alpha+1-n}\mathbf{H}](\mathbf{d}-\mathbf{c}), \\
&= \frac{1}{2\alpha+2}(1 + \frac{n}{\alpha+1-n})(d-c)\mathbf{e}, \\
&= \frac{d-c}{2\alpha+2-2n}\mathbf{e}.
\end{aligned}
$$

From Equation (4.9),

$$
\begin{aligned}
Z_{SO} &= \frac{1}{4}(\mathbf{d}-\mathbf{c})^T\mathbf{B}^{-1}(\mathbf{d}-\mathbf{c}), \\
&= \frac{1}{4}(\mathbf{d}-\mathbf{c})^T\frac{1}{\alpha+1}[\mathbf{I} + \frac{1}{\alpha+1-n}\mathbf{H}](\mathbf{d}-\mathbf{c}), \\
&= \frac{1}{4}(\mathbf{d}-\mathbf{c})^T\frac{1}{\alpha+1}(1 + \frac{n}{\alpha+1-n})(d-c)\mathbf{e}, \\
&= \frac{1}{4}\frac{d-c}{\alpha+1-n}(\mathbf{d}-\mathbf{c})^T\mathbf{e}, \\
&= \frac{1}{4}\frac{d-c}{\alpha+1-n}(\mathbf{d}-\mathbf{c})^T\mathbf{e}, \\
&= \frac{n(d-c)^2}{4\alpha+4-4n}.
\end{aligned}
$$

Following directly from Equation (4.11),

$$\mathbf{p_{SO}} = \frac{d+c}{2}\mathbf{e}.$$

$\blacksquare$

Remark The optimal profit generated under system optimization is independent of the wholesale price, $\mathbf{w_{SO}}$.

4.5.4 Analysis of Loss of Coordination under Uniform Demand

We will study the efficency of the system under uniform demand by analyzing the loss of coordination. We will first give the expression for the loss of coordination in the following theorem:

Theorem 4.5.6 *Under Assumption 3.7.1, 3.7.2 and 3.7.3, the loss of coordination under a uniform demand function in a Cournot competition with complement products is*

$$LOC = \frac{(3-r)(1-r)}{(2-r)^2},$$

where $r = \dfrac{n-1}{\alpha}$.

Proof From Proposition 4.17 and Proposition 4.20, the loss of coordination is

$$\begin{aligned} LOC &= \frac{Z_{UO}}{Z_{SO}}, \\ &= \frac{3\alpha+1-n}{4(2\alpha+1-n)^2}(4\alpha+4-4n), \\ &= \frac{(3\alpha+1-n)(\alpha+1-n)}{(2\alpha+1-n)^2}, \\ &= \frac{(3+\frac{1-n}{\alpha})(1+\frac{1-n}{\alpha})}{(2+\frac{1-n}{\alpha})^2}, \\ &= \frac{(3-r)(1-r)}{(2-r)^2}. \end{aligned}$$

∎

Remark The uniform demand model gives the worst case scenario for the loss of coordination in a Cournot competition with complement products, as deduced from the bound in Inequality (4.16) and Theorem 4.5.6. This implies that the lower bound in Inequality (4.16) is tight. The consequence is that in many real world instances, where we have a non-uniform setting, the performance under user optimization will be even closer to that under system optimization.

Discussions

Figure 4-1 shows the loss of coordination incurred by the supply chain under this model. There are three key observations regarding the loss of coordination in Theorem 4.5.6 as illustrated in Figure 4-1.

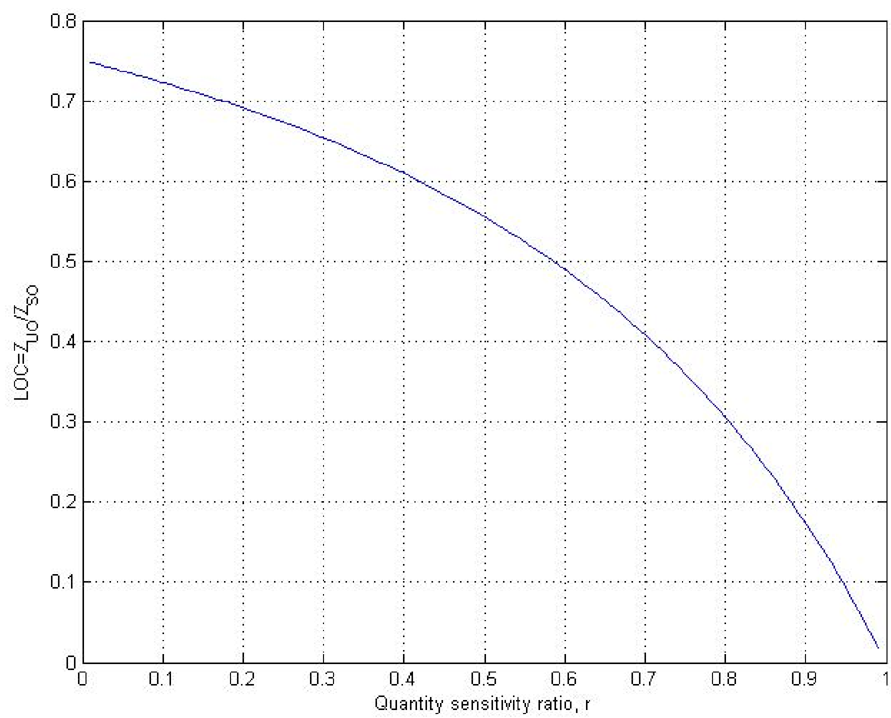

Figure 4-1: The loss of coordination in a Cournot competition with complement products and uniform demand.

1. As r tends to 1, the loss of coordination tends to zero. Under competition, retailers have incentives to lower production to keep their prices high. However, system optimality is achieved when every retailer increase their production. The parameter r measures the degree of complementary of the products. Under the extreme scenario when $r = 1$, and market clearing prices are not affected by an increase in production by every retailer. This occurs when products are highly complementary, and an increase in demand for one retailer leads to an increase in demand of the same quantity for the competitors. Coordinated response under system optimization will then require all retailers to increase production indefinitely, which leads to extremely high profits under system optimization, driving LOC to zero.

2. The loss of coordination in Equation (4.5.6) is strictly decreasing with respect to r. By Assumption 4.2.4 and Assumption 4.2.5, we can deduce that $0 \leq \alpha \leq n - 1$. Hence, $0 \leq r \leq 1$. Therefore, lower and upper bounds for the loss of coordination are attained when r takes values 1 and 0 respectively. We thus have
$$0 \leq LOC \leq \frac{3}{4},$$
which is also illustrated by Figure 4-1.

3. The loss of coordination is a concave function with respect to r. As such, it decreases slowly for small r and decreases more rapidly for large r. For instance, the loss of coordination stays within
$$0.6 \leq LOC \leq 0.75 \quad \text{for} \quad 0 \leq r \leq 0.419.$$
Intuitively, for low quantity sensitivity indicated by low r, market clearing prices are not significantly affected by quantity changes. Therefore, the total profit generated under user optimization remains relatively high despite a lack of coordination. When quantity sensitivity is high, a small increase in production quantity results in a large decrease in market clearing prices. In this situation,

a lack or coordination is costly, resulting in nearly zero profits for the supplier and all retailers.

4.6 Tightness of Bounds

We analyze the tightness of the lower bounds derived in the earlier sections by varying vector $\mathbf{d}$ with either n or $r_{max}(\mathbf{B})$. We will evaluate the tightness of the three lower bounds, which are in terms of $\lambda_{min}(\mathbf{G})$, $r_{max}(\mathbf{G})$ and $r_{max}(\mathbf{B})$, denoted by $LOC(\lambda_{min}(\mathbf{G}))$, $LOC(r_{max}(\mathbf{G}))$ and $LOC(r_{max}(\mathbf{B}))$ respectively.

In the first set of simulations, we generate 10000 random instances of matrix $\mathbf{B}$, which satisfies the assumptions in Section 4.2.1, for each n which varies from 2 to 20. In the second set of simulations, 1000 random instances of matrix $\mathbf{B}$ are generated for each $r_{max}(\mathbf{B})$ in steps of 0.01 from 0 to 1. We then obtain the averages of the loss of coordination and their bounds from these random instances. In both sets of simulations, we consider three scenarios for vector $\mathbf{d}$:

1. $\mathbf{d} = \mathbf{c} + \mathbf{r}$, where $\mathbf{r}$ is a random vector of mean one.

2. $\mathbf{d} = \mathbf{c} + \mathbf{k}$, where $\mathbf{k}$ is a constant vector of ones.

3. $\mathbf{d} = k(\mathbf{B} + \mathbf{\Gamma})\mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{v} + \mathbf{c}$, where $\mathbf{v}$ is the eigenvector corresponding to $\lambda_{min}(\mathbf{G})$, and $k \in \Re$.

Discussion

1. The bound in terms of $\lambda_{min}(\mathbf{G})$ is the tightest among the three lower bounds. In the scenario when $\mathbf{d} = k(\mathbf{B} + \mathbf{\Gamma})\mathbf{\Gamma}^{-\frac{1}{2}}\mathbf{v} + \mathbf{c}$, where $\mathbf{v}$ is the eigenvector corresponding to $\lambda_{min}(\mathbf{G})$, this lower bound is exactly equal to the actual loss of coordination, as stated in remark following Theorem 4.4.3.

2. The bound in terms of $\lambda_{min}(\mathbf{G})$ 'almost' coincide with the actual loss of coordination when $\mathbf{d} = \mathbf{c} + \mathbf{k}$, where $\mathbf{k}$ is a constant vector of ones. This happens when there is no quality differences among the retailers.

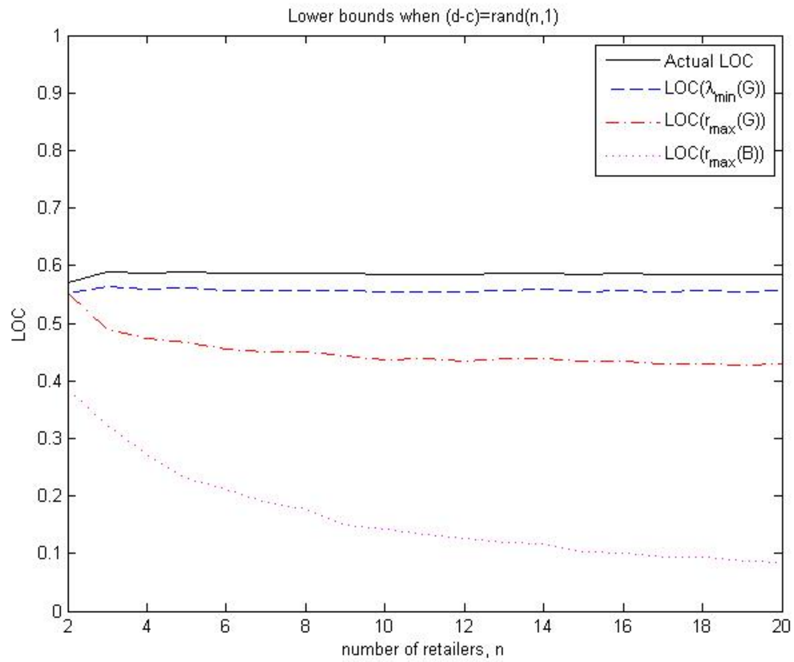

Figure 4-2: Lower Bounds with varying number of retailers, when $\mathbf{d} = \mathbf{c} + \mathbf{r}$, where $\mathbf{r}$ is a random vector of mean one.

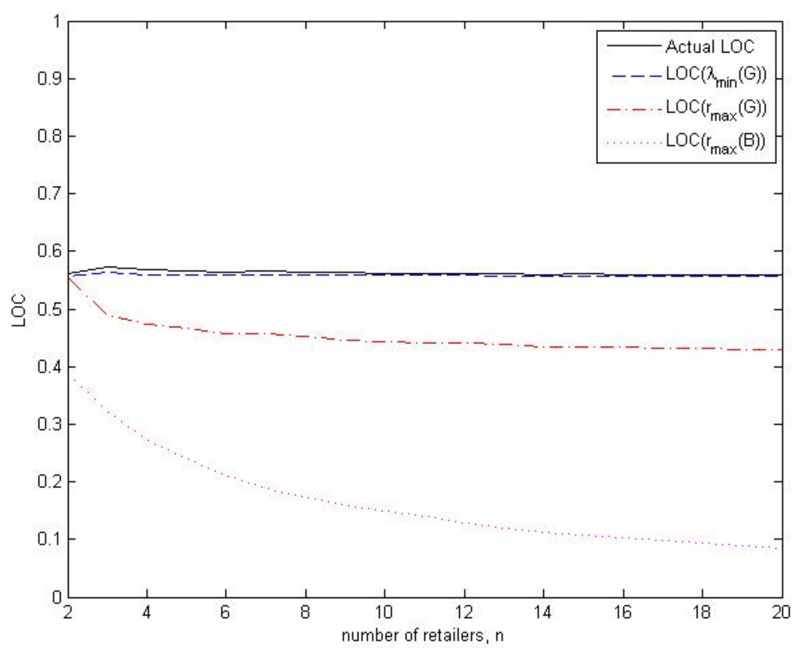

Figure 4-3: Lower Bounds with varying number of retailers, when $\mathbf{d} = \mathbf{c} + \mathbf{k}$, where $\mathbf{k}$ is a constant vector of ones.

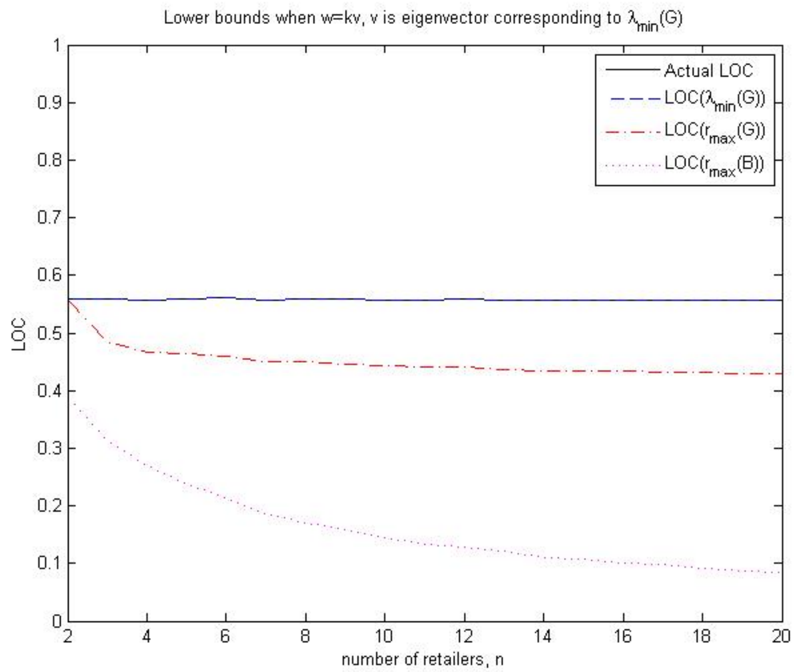

Figure 4-4: Lower Bounds with varying number of retailers, when $\mathbf{w} = \mathbf{v}$, where $\mathbf{v}$ is the eigenvector corresponding to $\lambda_{min}(\mathbf{G})$.

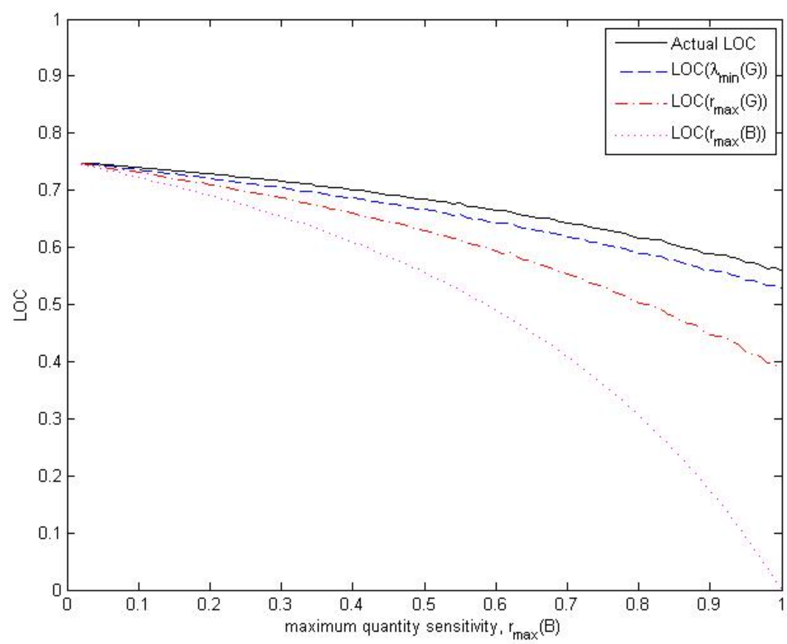

Figure 4-5: Lower Bounds with varying $r_{max}(\mathbf{B})$, when $\mathbf{d} = \mathbf{c}+\mathbf{r}$, where $\mathbf{r}$ is a random vector of mean one.

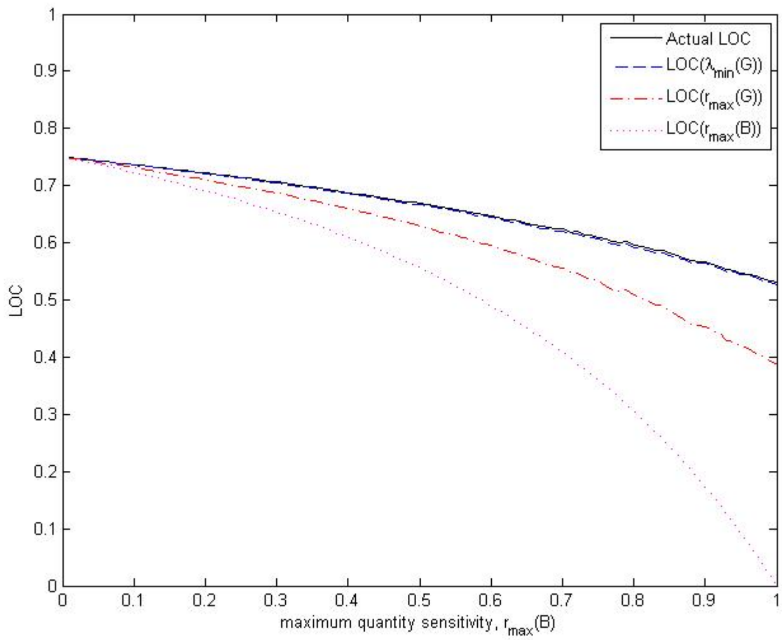

Figure 4-6: Lower Bounds with varying $r_{max}(\mathbf{B})$, when $\mathbf{d} = \mathbf{c} + \mathbf{k}$, where $\mathbf{k}$ is a constant vector of ones.

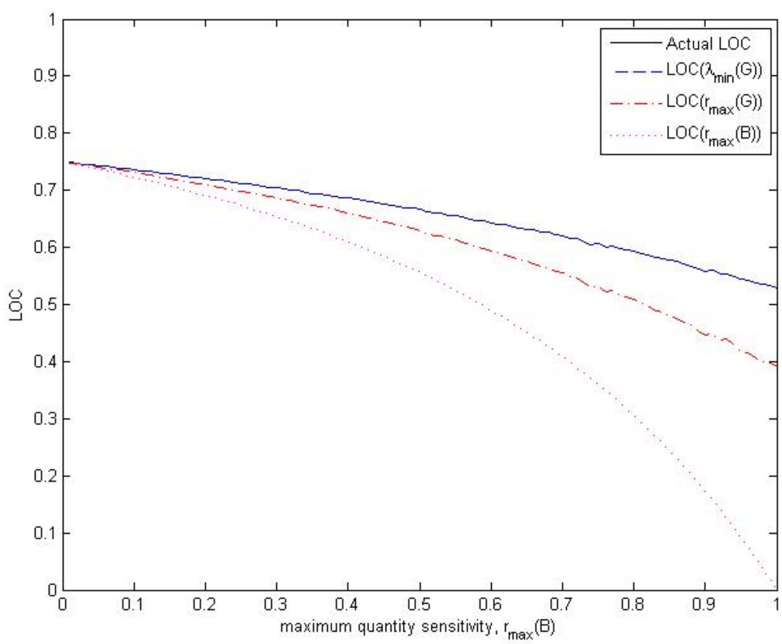

Figure 4-7: Lower Bounds with varying $r_{max}(\mathbf{B})$, when $\mathbf{w} = \mathbf{v}$, where $\mathbf{v}$ is the eigenvector corresponding to $\lambda_{min}(\mathbf{G})$.

3. The bound in terms of $r_{max}(\mathbf{G})$ is not as tight as the bound in terms of $\lambda_{min}(\mathbf{G})$. For example, when the average LOC is about 56%, the bound in terms of $\lambda_{min}(\mathbf{G})$ gives a value of 'almost' 56%, while the bound in terms of $r_{max}(\mathbf{G})$ gives a value of about 42%. However, it is much easier to compute the this lower bound, which is in terms of the diagonal dominance of matrix $\mathbf{G}$.

4. The actual loss of coordination is much higher than the bound in terms of $r_{max}(\mathbf{B})$, which coincides with the actual loss of coordination under uniform demand. This shows that although the loss of coordination under uniform demand gives a lower bound for loss of coordination under a general setting, the actual performance is much better than expected. As shown in the figures, the average loss of coordination is consistently above 50%, even though $LOC \rightarrow 0$ when $r_{max}(\mathbf{B}) \rightarrow 1$ under uniform demand.

Chapter 5

Multinomial Logit Demand

5.1 Overview and Main Contributions

In this chapter, we analyze the loss of profit due to lack of coordination (we refer to this as loss of coordination) in a single-tier multi-retailer Bertrand oligopoly market, where retailers compete by deciding their selling price. This model considers an Multinomial Logit (MNL) demand function which is derived from a probabilistic consumer utility function. The MNL demand belongs to an important class of discrete choice model, and it is a very valuable tool for microeconometric analysis of choice behaviour today. As a special case, we also consider a uniform MNL demand function, when all retailers encounter identical demand.

We evaluate the loss of coordination to measure the efficiency of the retailers under competition, computed as the ratio of the total profit (i.e., sum of all the retailers' proft) generated under competition (user optimization) and under coordination (system optimization). We then propose a lower bound for this loss of coordination to quantify the efficiency of these retailers under competition. As a special case, we consider a symmetric setting where all retailers encounter identical marginal costs, quality differences of products among competitors and variances in the probabilistic component of the consumers' utility function. Under the symmetric setting, we present three lower bounds on the loss of coordination. Finally, we conduct simulations to evaluate the tightness of these bounds and discuss some insights on the loss of coordination under the multinomial logit demand.

We find that profits and the loss of coordination are dependent on the number of retailers and predictability of consumer behaviour. Higher predictability of consumer behaviour increases profits both under coordination and under competition, but predictability benefits coordination more than competition. Increasing the number of retailers decreases profits under competition, but increases profit under coordination. As such, the efficiency of competition deteoriates with increasing number of retailers and predictability of consumer behaviour.

The structure of the remainder of this chapter is as follows. Section 5.2 provides the groundwork for this chapter describing the model used in this chapter, including the central concept of Nash equilibrium, user optimization, system optimization, and the loss of coordination. This section also contains the associated notations and assumptions imposed on our analysis. In Section 5.3, we considered the general case when demand is asymmetric (that is, retailers encounter different marginal costs, offers product of different quality from competitors, and different variances in consumer behaviour). We derive the total profit under user optimization and system optimization, write the loss of coordination in terms of the optimal solutions of the two optimization problems, and present a lower bound for the loss of coordination. In the following Section 5.4, we consider a setting where retailers encounter identical marginal costs, quality differences and consumer behaviour indicated by the same variances. We derive the optimality conditions, present the loss of coordination and propose three lower bounds for the loss of coordination. Finally, in Section 5.5, we present simulation results to evaluate the tightness of the bounds on the optimal profits in Subsection 5.5.1 and compare the bounds on the loss of coordination in Subsection 5.5.2. We conclude this section with a discussion on the behaviour of optimal profits and loss of coordination with varying number of retailers, marginal costs and predictability of consumer behaviour.

5.2 Model Description

We consider a single-tier supply chain setting, where multiple retailers of differentiated products compete under a multinomial logit (MNL) demand function by deciding the retail prices. The MNL demand function arises from a discrete choice model employing discrete choice, that employs discrete choice analysis to model the probability of a consumer choosing a particular retailer for their product. The consumer utility has a deterministic component (which measures the quality differences of the product) and a probabilistic component. In a discrete choice model, a consumer always makes a decision to receive the highest utility. We refer the reader to Anderson, de Palma and Thisse (1992) for more information on the MNL demand function.

In this oligopoly market of n retailers, we denote the demand of retailer i ($i = 1, 2, ..., n$) by q_i and let vector $\mathbf{q} = (q_1, ..., q_n)^T$. Similarly, let $\mathbf{p}$ be the vector selling prices, p_i, and $\mathbf{c}$ be the vector of constant marginal production costs, c_i.

Let Z_{R_i} be the profit of retailer i, and Z be the sum of all retailers' profit. Let the equilibrium selling prices, demand and total profit under competition (user optimization) be denoted by $\mathbf{p_{UO}}$, $\mathbf{q_{UO}}$ and Z_{UO} respectively. Let the optimal selling prices, demand and total profit under coordination (system optimization) be denoted by $\mathbf{p_{SO}}$, $\mathbf{q_{SO}}$ and Z_{SO} respectively

Our analysis is restricted to models that satisfy the following assumptions:

Assumption 5.2.1 *Each retailer has unlimited production capacity.*

Therefore, each retailer is able to meet any market demand.

Assumption 5.2.2 *Each retailer has constant marginal cost of production,* $\mathbf{c} = (c_1, c_2, ..., c_n)$.

We use the MNL demand function given by the multinomial logit (MNL) model described in Talluri and van Ryzin (2004).

$$q_i(\mathbf{p}) = \frac{Ke^{(d_i - p_i)/\alpha_i}}{A + \sum_{j=1}^{n} e^{(d_j - p_j)/\alpha_j}}, \tag{5.1}$$

where d_i is the deterministic component of the consumer utility, α_i is the variance of the probabilistic component, K is a large positive number denoting the market size, and A is an outside alternative.

Without loss of generality, set $A = 1$. We also let $v_i = e^{(d_i - c_i)/\alpha_i}$, $x_i = \frac{p_i - c_i}{\alpha_i}$ be the scaled profit margin for retailer i. and $\mathbf{x}$ to denote the vector of scaled profit margins. The MNL demand function can be written as

$$q_i(\mathbf{p}) = \frac{K v_i e^{-x_i}}{1 + \sum_{j=1}^{n} v_j e^{-x_j}}, \quad i = 1, ..., n. \tag{5.2}$$

Under user optimization, the retailers decide on the prices to charge the consumers as a best response to other retailers' equilibrium pricing policies. Each retailer is assumed to be rational and selfish, optimizing profits only for themselves. Nash Equilibrium is reached when no single retailer can increase its profit by unilaterally changing its pricing policy. This implies that each retailer sets its equilibrium prices as the best response to the equilibrium prices of its competitors.

For each retailer i, given the competitors' equilibrium prices denoted by the vector $\mathbf{p_{UO,-i}}$, the retailer's pricing policy is obtained by solving the optimization problem $\mathbf{UO_{R_i}}$ described as follows:

$$\begin{aligned} \mathbf{UO_{R_i}}: \quad & \max_{p_i} \quad (p_i - c_i).q_i(p_i, \mathbf{p_{UO,-i}}), \\ & \text{s.t.} \quad q_i \geq 0. \end{aligned} \tag{5.3}$$

Let $Z_{R_i}^{UO}$ denote the profit of retailer i obtained from Optimization Problem (5.3). The total profit (sum of profit of all retailers) under user optimization, Z_{UO}, is given by

$$Z_{UO} = \sum_{i=1}^{n} Z_{R_i}^{UO}.$$

Under system optimization, a central authority is coordinating all decisions, optimizing the total profit. The central authority makes decisions on the selling prices and forces all retailers to comply. Coordination is attained by solving the following optimization problem, which determines the pricing policy that maximizes total profit.

$$\begin{aligned} \textbf{SO:} \quad \max_{\mathbf{p_{SO}}} \quad & \sum_{i=1}^{n} (p_i - c_i).q_i(p_i, \mathbf{p_{UO,-i}}), \\ \text{s.t.} \quad & \mathbf{q_{SO}} \geq 0. \end{aligned} \tag{5.4}$$

Let Z_{SO} denote the optimal combined profit obtained by solving the above optimization problem.

The loss of coordination, LOC, measures the loss of the total profit under competition, computed as the ratio of the total profit generated under user optimization and under system optimization. That is,

$$LOC = \frac{Z_{UO}}{Z_{SO}}. \tag{5.5}$$

5.3 Loss of Coordination: Asymmetric Firms

This is the asymmetric setting, that is, where each parameter v_i, c_i and α_i is not necessarily the same for different products $i = 1, ..., n$. We will first derive the loss of coordination and total profit under user and system optimization in Proposition 5.3.1, and present a lower bound for the loss of coordination in Theorem 5.3.3.

Proposition 5.3.1 *Under asymmetric multinomial logit demand, the total profit under user optimization and system optimization are, respectively,*

$$Z_{UO} = \sum_{i=1}^{n} K\alpha_i(x_i^{UO} - 1). \tag{5.6}$$

$$Z_{SO} = K\alpha_i(x_i^{SO} - 1). \tag{5.7}$$

Consequently, the loss of coordination is

$$LOC = \frac{\sum_{i=1}^{n} \alpha_i (x_i^{UO} - 1)}{\alpha_i (x_i^{SO} - 1)}. \tag{5.8}$$

Proof The profit for retailer i is

$$\begin{aligned} Z_i &= q_i(\mathbf{p})(p_i - c_i), \\ &= \frac{K v_i e^{-x_i}}{1 + \sum_{j=1}^{n} v_j e^{-x_j}} (\alpha_i x_i), \\ &= \frac{K v_i \alpha_i x_i e^{-x_i}}{1 + \sum_{j=1}^{n} v_j e^{-x_j}}, \end{aligned} \tag{5.9}$$

Therefore, Optimization Probelm 5.3 is equivalent to

$$\begin{aligned} \mathbf{UO_i}: \quad & \max_{x_i} \quad \frac{K v_i \alpha_i x_i e^{-x_i}}{1 + \sum_{j=1}^{n} v_j e^{-x_j}}, \\ & \text{s.t.} \quad q_i \geq 0. \end{aligned} \tag{5.10}$$

We relax the non-negativity constraints in Optimization Problem (5.10) to determine the equilibrium pricing policy for retailer i, which is achieved when

$$\frac{\partial Z_i}{\partial x_i} = \frac{K \alpha_i v_i (e^{-x_i} - x_i e^{-x_i})(1 + \sum_{j=1}^{n} v_j e^{-x_j}) - (K \alpha_i v_i x_i e^{-x_i})(-v_i e^{-x_i})}{(1 + \sum_{j=1}^{n} v_j e^{-x_j})^2}.$$

At optimality, $\dfrac{\partial Z_i}{\partial x_i} = 0$. Therefore,

$$\alpha_i v_i (e^{-x_i^{UO}} - x_i^{UO} e^{-x_i^{UO}})(1 + \sum_{j=1}^{n} v_j e^{-x_j^{UO}}) - (\alpha_i v_i x_i^{UO} e^{-x_i^{UO}})(-v_i e^{-x_i^{UO}}) = 0,$$

$$(e^{-x_i^{UO}} - x_i^{UO} e^{-x_i^{UO}})(1 + \sum_{j=1}^{n} v_j e^{-x_j^{UO}}) = (x_i^{UO} e^{-x_i^{UO}})(-v_i e^{-x_i^{UO}}),$$

$$(1 - x_i^{UO})(1 + \sum_{j=1}^{n} v_j e^{-x_j^{UO}}) = -v_i x_i^{UO} e^{-x_i^{UO}}, \tag{5.11}$$

$$x_i^{UO} - 1 = \frac{v_i x_i^{UO} e^{-x_i^{UO}}}{1 + \sum_{j=1}^{n} v_j e^{-x_j^{UO}}}, \tag{5.12}$$

We will now derive the optimality conditions and profit under system optimization. From Equation (5.9), the total profit of all the retailers is

$$Z = \frac{K \sum_{i=1}^{n} v_i \alpha_i x_i e^{-x_i}}{1 + \sum_{j=1}^{n} v_j e^{-x_j}}, \tag{5.13}$$

Therefore, Optimization Problem (5.4) is equivalent to

$$\begin{aligned} \textbf{SO:} \quad \max_{\mathbf{x}} \quad & \frac{K \sum_{i=1}^{n} v_i \alpha_i x_i e^{-x_i}}{1 + \sum_{j=1}^{n} v_j e^{-x_j}}, \\ \text{s.t.} \quad & \mathbf{q_{SO}} \geq 0. \end{aligned} \tag{5.14}$$

We relax the non-negativity constraints in Optimization Problem (5.14) to determine the optimal pricing policy for retailer i, which is achieved when

$$\frac{\partial Z}{\partial x_i} = \frac{K\alpha_i v_i(e^{-x_i} - x_i e^{-x_i})(1+\sum_{j=1}^{n} v_j e^{-x_j}) - (K\sum_{j=1}^{n}\alpha_j v_j x_j e^{-x_j})(-v_i e^{-x_i})}{(1+\sum_{j=1}^{n} v_j e^{-x_j})^2} \tag{5.15}$$

At optimality, $\frac{\partial Z}{\partial x_i} = 0$. Therefore,

$$\alpha_i v_i(e^{-x_i^{SO}} - x_i^{SO} e^{-x_i^{SO}})(1+\sum_{j=1}^{n} v_j e^{-x_j^{SO}}) - (\sum_{j=1}^{n}\alpha_j v_j x_j^{SO} e^{-x_j^{SO}})(-v_i e^{-x_i^{SO}}) = 0,$$

$$(1 - x_i^{SO})\alpha_i v_i e^{-x_i^{SO}}(1+\sum_{j=1}^{n} v_j e^{-x_j^{SO}}) = -v_i e^{-x_i^{SO}}\sum_{j=1}^{n}\alpha_j v_j x_j^{SO} e^{-x_j^{SO}}, \tag{5.16}$$

$$x_i^{SO} - 1 = \frac{v_i e^{-x_i^{SO}}\sum_{j=1}^{n}\alpha_j v_j x_j^{SO} e^{-x_j^{SO}}}{\alpha_i v_i e^{-x_i^{SO}}(1+\sum_{j=1}^{n} v_j e^{-x_j^{SO}})},$$

$$K\alpha_i(x_i^{SO} - 1) = \frac{K\sum_{j=1}^{n}\alpha_j v_j x_j^{SO} e^{-x_j^{SO}}}{1+\sum_{j=1}^{n} v_j e^{-x_j^{SO}}}, \tag{5.17}$$

From Equations (5.13) and (5.12), the total profit under user optimization is

$$Z_{UO} = \frac{K\sum_{i=1}^{n} v_i \alpha_i x_i^{UO} e^{-x_i^{UO}}}{1+\sum_{j=1}^{n} v_j e^{-x_j^{UO}}}, \tag{5.18}$$

$$= K\sum_{i=1}^{n}\alpha_i(x_i^{UO} - 1). \tag{5.19}$$

From Equations (5.13) and (5.17), the total profit under system optimization is

$$Z_{SO} = \frac{K \sum_{j=1}^{n} \alpha_j v_j x_j^{SO} e^{-x_j^{SO}}}{1 + \sum_{j=1}^{n} v_j e^{-x_j^{SO}}}, \tag{5.20}$$

$$= K\alpha_i (x_i^{SO} - 1). \tag{5.21}$$

Finally, with Equations (5.21) and Equation (5.19), we can write the loss of coordination in terms of the optimal solutions as follows:

$$LOC = \frac{\sum_{i=1}^{n} \alpha_i (x_i^{UO} - 1)}{\alpha_i (x_i^{SO} - 1)}. \tag{5.22}$$

$\blacksquare$

Remark The optimality conditions in Equations (5.12) and (5.17) imply that $x_i^{UO} \geq 1$ and $x_i^{SO} \geq 1$.

We will now derive a lower bound for the loss of coordination. In the derivation of this bound, we need several results on the bounds of the optimal solutions, x_i^{SO} and x_i^{UO}, which will be presented in the follow lemma:

Lemma 5.3.2 *The optimal solutions, x_i^{UO} and x_i^{SO}, of the user and system optimization problems under asymmetric multinomial logit demand are bounded by the following:*

$$x_i^{SO} \geq 1 + \frac{1}{\alpha_i} \sum_{j=1}^{n} \alpha_j (x_j^{UO} - 1), \tag{5.23}$$

$$x_i^{SO} \geq x_i^{UO}, \tag{5.24}$$

$$x_j^{UO} e^{-x_j^{UO}} \geq x_j^{SO} e^{-x_j^{SO}}, \tag{5.25}$$

$$x_i^{SO} \leq 1 + \frac{1}{\alpha_i} \sum_{j=1}^{n} \alpha_j v_j x_j^{UO} e^{-x_j^{UO}}. \tag{5.26}$$

Proof From Equation (5.8) and the fact that $LOC \leq 1$,

$$\frac{\sum_{j=1}^{n} \alpha_j (x_j^{UO} - 1)}{\alpha_i (x_i^{SO} - 1)} \leq 1,$$

$$\sum_{j=1}^{n} \alpha_j (x_j^{UO} - 1) \leq \alpha_i (x_i^{SO} - 1).$$

Therefore, we arrive at Inequality (5.23):

$$x_i^{SO} \geq 1 + \frac{1}{\alpha_i} \sum_{j=1}^{n} \alpha_j (x_j^{UO} - 1).$$

Inequality (5.24) follows from Inequality (5.23) as shown:

$$\begin{aligned} x_i^{SO} &\geq 1 + \frac{1}{\alpha_i} (\alpha_i (x_i^{UO} - 1) + \sum_{j \neq i}^{n} \alpha_j (x_j^{UO} - 1)), \\ &= x_i^{UO} + \sum_{j \neq i}^{n} \alpha_j (x_j^{UO} - 1)), \\ &\geq x_i^{UO}. \end{aligned}$$

The third bound in Inequality (5.25) follows directly from Inequality (5.24) since xe^{-x} is decreasing in x, for $x \geq 1$.

From Inequalities (5.24), (5.25) and the optimality conditions in Equation (5.20) and (5.21),

$$
\begin{aligned}
K\alpha_i(x_i^{SO}-1) &= \frac{K\sum_{j=1}^{n}\alpha_j v_j x_j^{SO}e^{-x_j^{SO}}}{1+\sum_{j=1}^{n}v_j e^{-x_j^{SO}}}, \\
&\leq \frac{K\sum_{j=1}^{n}\alpha_j v_j x_j^{UO}e^{-x_j^{UO}}}{1+\sum_{j=1}^{n}v_j e^{-x_j^{SO}}}, \\
&\leq K\sum_{j=1}^{n}\alpha_j v_j x_j^{UO}e^{-x_j^{UO}}.
\end{aligned}
$$

Therefore, we arrive at Inequality (5.26)

$$x_i^{SO} \leq 1+\frac{1}{\alpha_i}\sum_{j=1}^{n}\alpha_j v_j x_j^{UO}e^{-x_j^{UO}}.$$

■

We are now ready to derive a lower bound for the loss of coordination under asymmetric multinomial logit demand

Theorem 5.3.3 *A lower bound for the loss of coordination under asymmetric multinomial logit demand is given by*

$$LOC \geq 1-\frac{\sum_{i=1}^{n}(\frac{v_i}{\alpha_i}\sum_{j=1}^{n}\alpha_j v_j)}{e^2+e\sum_{j=1}^{n}v_j}. \tag{5.27}$$

Proof From Equation (5.5), (5.20) and (5.18), the loss of coordination is

$$LOC = \frac{K\sum_{j=1}^{n}\alpha_j v_j x_j^{UO} e^{-x_j^{UO}}}{1+\sum_{j=1}^{n} v_j e^{-x_j^{UO}}} \cdot \frac{1+\sum_{j=1}^{n} v_j e^{-x_j^{SO}}}{K\sum_{j=1}^{n}\alpha_j v_j x_j^{SO} e^{-x_j^{SO}}}.$$

From Inequality (5.24), $x_i^{SO} e^{-x_i^{SO}} \leq x_i^{UO} e^{-x_i^{UO}}$. Therefore,

$$\begin{aligned} LOC &\geq \frac{K\sum_{j=1}^{n}\alpha_j v_j x_j^{SO} e^{-x_j^{SO}}}{1+\sum_{j=1}^{n} v_j e^{-x_j^{UO}}} \cdot \frac{1+\sum_{j=1}^{n} v_j e^{-x_j^{SO}}}{K\sum_{j=1}^{n}\alpha_j v_j x_j^{SO} e^{-x_j^{SO}}}, \\ &= \frac{1+\sum_{j=1}^{n} v_j e^{-x_j^{SO}}}{1+\sum_{j=1}^{n} v_j e^{-x_j^{UO}}}. \end{aligned}$$

By convexity of e^{-x},

$$\begin{aligned} e^{-x_j^{SO}} &\geq e^{-x_j^{UO}} - e^{-x_j^{UO}}(x_j^{SO} - x_j^{UO}), \\ &= e^{-x_j^{UO}}(1 - x_j^{SO} + x_j^{UO}). \end{aligned}$$

It follows that

$$\begin{aligned} LOC &\geq \frac{1+\sum_{j=1}^{n} v_j e^{-x_j^{UO}}(1 - x_j^{SO} + x_j^{UO})}{1+\sum_{j=1}^{n} v_j e^{-x_j^{UO}}}. \\ &= 1 - \frac{\sum_{j=1}^{n} v_j e^{-x_j^{UO}}(x_j^{SO} - x_j^{UO})}{1+\sum_{j=1}^{n} v_j e^{-x_j^{UO}}}. \end{aligned}$$

From the upper bound for x_j^{SO} in Inequality (5.26), we arrive at

$$LOC \geq 1 - \frac{\sum_{i=1}^{n} v_i e^{-x_i^{UO}} (1 + \frac{1}{\alpha_i} \sum_{j=1}^{n} \alpha_j v_j x_j^{UO} e^{-x_j^{UO}} - x_i^{UO})}{1 + \sum_{j=1}^{n} v_j e^{-x_j^{UO}}}. \tag{5.28}$$

Since the RHS is increasing with respect to x_i^{UO}, and $x_i^{UO} \geq 1$, a lower bound for the LOC in terms of the constant parameters, α's and v's, is

$$\begin{aligned}
LOC &\geq 1 - \frac{\sum_{i=1}^{n} v_i e^{-1} (1 + \frac{1}{\alpha_i} \sum_{j=1}^{n} \alpha_j v_j e^{-1} - 1)}{1 + \sum_{j=1}^{n} v_j e^{-1}}, \\
&= 1 - \frac{\sum_{i=1}^{n} e^{-1} (\frac{v_i}{\alpha_i} \sum_{j=1}^{n} \alpha_j v_j e^{-1})}{1 + \sum_{j=1}^{n} v_j e^{-1}}, \\
&= 1 - \frac{\sum_{i=1}^{n} (\frac{v_i}{\alpha_i} \sum_{j=1}^{n} \alpha_j v_j)}{e^2 + e \sum_{j=1}^{n} v_j}.
\end{aligned}$$

∎

Discussion To obtain some insights into this lower bound, we performed simulations in a symmetric setting where $\alpha_i = \alpha$ and $v_i = v$ for all i, and evaluated the tightness of the lower bound under this setting. The results of the simulations are shown in Figures 5-1, 5-2 and 5-3.

1. We observe that both the LOC and the lower bound for the LOC decreases as n and v increases.

2. The lower bound is 'rather' tight when v is small. For example, when $v = 0.1$, the actual LOC is above 98%, while the lower bound is always above 91%.

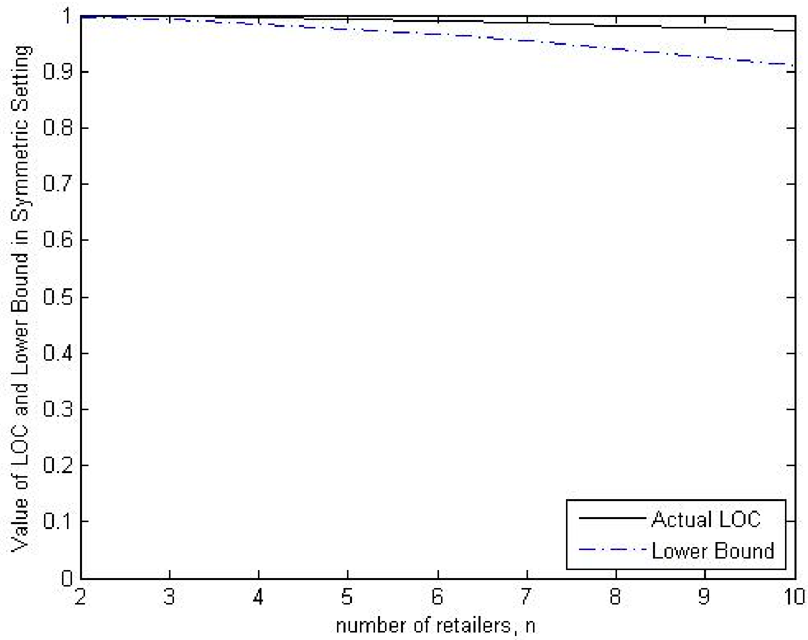

Figure 5-1: Lower Bound for LOC in symmetric setting with varying number of retailers, when $v = 0.1$.

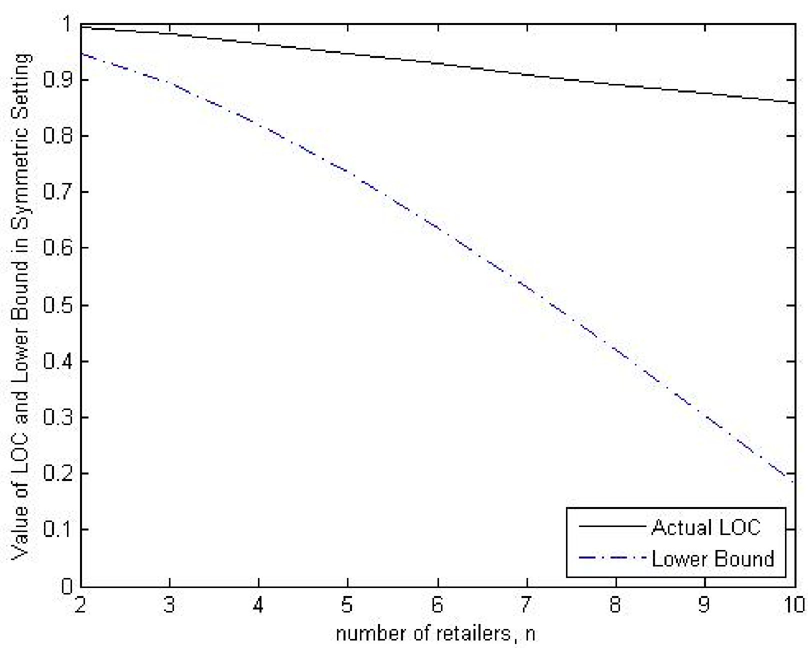

Figure 5-2: Lower Bound for LOC in symmetric setting with varying number of retailers, when $v = 0.4$.

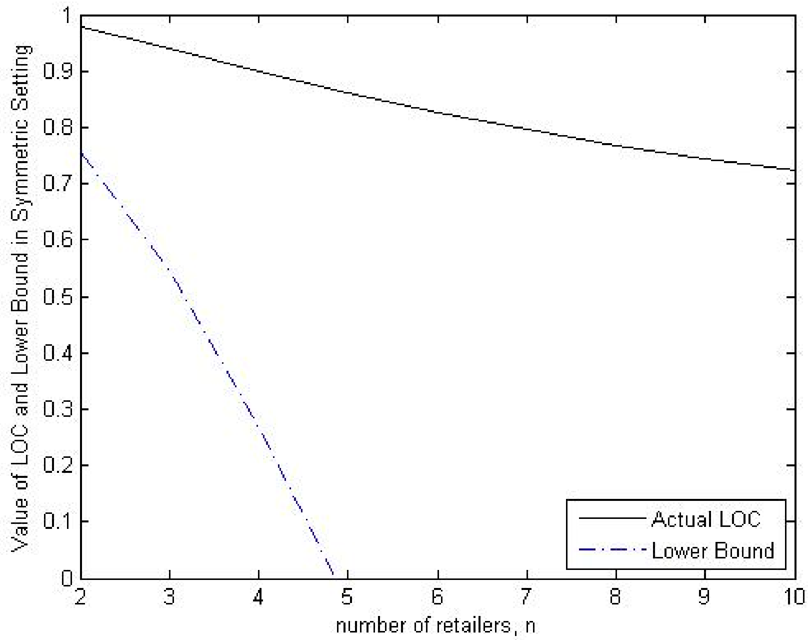

Figure 5-3: Lower Bound for LOC in symmetric setting with varying number of retailers, when $v = 1$.

3. The quality of the bound deteoriates when v is larger, namely, beyond $v = 0.4$. As an example, the lower bound is negative when $n \geq 5$ for $v = 1$, which is redundant since it is known that $LOC \geq 0$. Since the lower bound is concave in n and v, the quality of the bound deteoriates quickly as n and v increases.

This motivates the derivation of another lower bound, which performs better under larger n and v. This lower bound, under the symmetric setting, will be presented in the following section.

5.4 Symmetric Retailers

We investigate the loss of coordination by considering the case when retailers are symmetric. Retailers are said to be symmetric under the following additional assumption:

Assumption 5.4.1 *The parameters c_i, d_i and α_i are equal across all retailers.*

This imply that $v_i = v$ for all retailers. This assumption holds if firms have equal marginal costs, c_i, equal values of the deterministic component, d_i, and equal variances, α_i, of the probabilistic component of the utility function.

In the following theorem, we derive the loss of coordination under the symmetric setting. Using the optimality conditions under user and system optimization, we will write the loss of coordination in two different ways.

Theorem 5.4.2 *The optimality conditions for the system optimization and user optimization under symmetric multinomial logit demand are*

$$x^{SO} - 1 = nve^{-x^{SO}}, \tag{5.29}$$

$$x^{UO} - 1 = \frac{v}{e^{x^{UO}} + (n-1)v}. \tag{5.30}$$

Therefore, the loss of coordination can be written as

$$LOC = \frac{n(x^{UO} - 1)}{x^{SO} - 1}, \tag{5.31}$$

$$LOC = \frac{e^{x^{SO}}}{e^{x^{UO}} + (n-1)v}. \tag{5.32}$$

Proof Under a symmetric setting, $\alpha_i = \alpha$, $v_i = v$, $x_i^{UO} = x^{UO}$ and $x_i^{SO} = x^{SO}$ for all $i = 1, 2, ..., n$. The optimality condition under system optimization is then a special case of Equation (5.16),

$$\alpha_i(1 - x_i^{SO})(1 + \sum_{j=1}^{n} v_j e^{-x_j^{SO}}) = -\sum_{j=1}^{n} \alpha_j v_j x_j^{SO} e^{-x_j^{SO}},$$

$$\alpha(1 - x^{SO})(1 + nve^{-x^{SO}}) = -n\alpha v x^{SO} e^{-x^{SO}},$$

$$x^{SO} - 1 = nve^{-x^{SO}}.$$

Under user optimization, the optimality condition can be obtained from Equation (5.11),

$$(1 - x_i^{UO})(1 + \sum_{j=1}^{n} v_j e^{-x_j^{UO}}) = -v_i x_i^{UO} e^{-x_i^{UO}},$$

$$(1 - x^{UO})(1 + nve^{-x^{UO}}) = -vx^{UO}e^{-x^{UO}},$$

$$1 + nve^{-x^{UO}} - x^{UO}(1 + (n-1)ve^{-x^{UO}}) = 0,$$

$$x^{UO} = \frac{e^{x^{UO}} + nv}{e^{x^{UO}} + (n-1)v},$$

$$x^{UO} - 1 = \frac{v}{e^{x^{UO}} + (n-1)v}.$$

The loss of coordination for the symmetric setting can be obtained as a special case of the asymmetric setting in Equation (5.8),

$$\begin{aligned} LOC &= \frac{\sum_{i=1}^{n} \alpha_i(x_i^{UO} - 1)}{\alpha_i(x_i^{SO} - 1)}, \\ &= \frac{n\alpha(x^{UO} - 1)}{\alpha(x^{SO} - 1)}, \\ &= \frac{n(x^{UO} - 1)}{x^{SO} - 1}. \end{aligned}$$

From Equation (5.31), and the optimality conditions in Equation (5.29) and (5.30), the loss of coordination can also be written as

$$\begin{aligned} LOC &= \frac{nv}{e^{x^{UO}} + (n-1)v} . \frac{1}{nve^{-x^{SO}}}, \\ &= \frac{e^{x^{SO}}}{e^{x^{UO}} + (n-1)v}. \end{aligned}$$

∎

Before deriving bounds on the loss of coordination, we will first present upper and lower bounds on both optimal solutions, $\mathbf{x^{SO}}$ and $\mathbf{x^{UO}}$.

Lemma 5.4.3 *Under the symmetric multinomial logit demand function, the optimal solutions, x^{UO} and x^{SO}, are bounded by*

$$k \leq x^{UO} \leq 1 + \frac{1}{n-1}, \tag{5.33}$$

$$1 - n + nk \leq x^{SO} \leq 1 + nvke^{-k}. \tag{5.34}$$

where $k = 1 + \dfrac{v}{e^{\frac{n}{n-1}} + (n-1)v}$, *a constant that depends only on* n *and* v.

Proof From the optimality condition in Equation (5.30),

$$x^{UO} - 1 = \frac{v}{e^{x^{UO}} + (n-1)v},$$

$$e^{x^{UO}} = \frac{v}{x^{UO} - 1} - (n-1)v.$$

Since $e^{x^{UO}} \geq 0$, it follows that

$$\frac{v}{x^{UO} - 1} \geq (n-1)v,$$

$$x^{UO} \leq 1 + \frac{1}{n-1}.$$

A lower bound for x^{UO} can be obtained from this upper bound as follows:

$$x^{UO} - 1 = \frac{v}{e^{x^{UO}} + (n-1)v} \geq \frac{v}{e^{1+\frac{1}{n-1}} + (n-1)v},$$

$$x^{UO} \geq 1 + \frac{v}{e^{\frac{n}{n-1}} + (n-1)v} = k.$$

We will now derive a lower bound for x^{SO} from the expression for the loss of coordination in Equation (5.31), the fact that $LOC \leq 1$, and the lower bound for x^{UO} in Inequality (5.33) as follows:

$$\begin{aligned} x^{SO} - 1 &\geq n(x^{UO} - 1), \qquad (5.35) \\ x^{SO} &\geq 1 - n + nx^{UO}, \\ &\geq 1 - n + nk. \end{aligned}$$

From the expressions for the total profits under system optimization obtained in

Equation (5.20) and (5.21),

$$\begin{aligned}
K\alpha(x^{SO}-1) &= \frac{Kn\alpha v x^{SO} e^{-x^{SO}}}{1+nve^{-x^{SO}}}, \\
&\leq \frac{Kn\alpha v x^{UO} e^{-x^{UO}}}{1+nve^{-x^{SO}}}, \quad \text{by Inequality (5.26)} \\
&\leq Kn\alpha v x^{UO} e^{-x^{UO}}.
\end{aligned}$$

It is also known that xe^{-x} is increasing in x, for $x \geq 1$. Therefore, an upper bound for x^{SO} is

$$\begin{aligned}
x^{SO} &\leq 1 + nvx^{UO}e^{-x^{UO}}, \\
&\leq 1 + nvke^{-k}.
\end{aligned}$$

∎

We are now ready to derive lower bounds for the loss of coordination under the symmetric setting in the following theorem:

Theorem 5.4.4 *Under the symmetric multinomial logit demand function, the loss of coordination is lower bounded by:*

$$LOC \geq \max\left\{\frac{e^{1-n+nk}}{e^k+(n-1)v}, \frac{1+nvke^{-k}-n^2v^2ke^{-2k}}{1+nve^{-k}}\right\}.$$

where $k = 1 + \dfrac{v}{e^{\frac{n}{n-1}}+(n-1)v}$, *a constant that depends only on* n *and* v.

Proof From Equation (5.32) and Inequality (5.35),

$$LOC = \frac{e^{x^{SO}}}{e^{x^{UO}}+(n-1)v} \geq \frac{e^{1-n+nx^{UO}}}{e^{x^{UO}}+(n-1)v}.$$

Since $\dfrac{e^{1-n+nx}}{e^x+(n-1)v}$ is increasing in x, a lower bound for the loss of coordination can be obtained from the lower bound in x^{UO} given in Inequality (5.33).

$$LOC \geq \frac{e^{1-n+nx^{UO}}}{e^{x^{UO}}+(n-1)v} \geq \frac{e^{1-n+nk}}{e^k+(n-1)v}. \tag{5.36}$$

Moreover, from Inequality (5.28),

$$
\begin{aligned}
LOC &\geq 1 - \frac{\displaystyle\sum_{i=1}^{n} v_i e^{-x_i^{UO}} (1 + \frac{1}{\alpha_i} \sum_{j=1}^{n} \alpha_j v_j x_j^{UO} e^{-x_j^{UO}} - x_i^{UO})}{\displaystyle 1 + \sum_{j=1}^{n} v_j e^{-x_j^{UO}}}, \\
&\geq 1 - \frac{nve^{-x^{UO}}(1 + nvx^{UO}e^{-x^{UO}} - x^{UO})}{1 + nve^{-x^{UO}}}, \\
&\geq \frac{1 + nvx^{UO}e^{-x^{UO}} - n^2v^2x^{UO}e^{-2x^{UO}}}{1 + nve^{-x^{UO}}}.
\end{aligned}
$$

Since $\dfrac{1 + nvxe^{-x} - n^2v^2xe^{-2x}}{1 + nve^{-x}}$ is increasing with respect to x, and $x^{UO} \geq k$ from Inequality (5.33),

$$
LOC \geq \frac{1 + nvke^{-k} - n^2v^2ke^{-2k}}{1 + nve^{-k}}. \tag{5.37}
$$

Since both Inequalities (5.36) and (5.37) are valid bounds for the loss of coordination, we can consolidate them and write the lower bound for the loss of coordination as follows:

$$
LOC \geq \max\left\{ \frac{e^{1-n+nk}}{e^k + (n-1)v}, \frac{1 + nvke^{-k} - n^2v^2ke^{-2k}}{1 + nve^{-k}} \right\}.
$$

∎

Remark In addition to the lower bounds given in Theorem 5.4.4, we also have five other lower bounds. We discuss them as follows:

1. A lower bound in terms of the optimal solution in system optimization, x^{SO}, is given by

$$
LOC \geq \frac{e^{x^{SO}}}{e^{x^{SO}} + (n-1)v}, \tag{5.38}
$$

   Since $x^{SO} \geq x^{UO}$ by Inequality (5.24), this be derived from the expression for the loss of coordination given in Equation (5.32),

$$
LOC = \frac{e^{x^{SO}}}{e^{x^{UO}} + (n-1)v}.
$$

2. A quick and neat lower bound is given by

$$LOC \geq \frac{e}{e + (n-1)v}.$$

This is obtained from Inequality (5.38), since $x_{SO} \geq 1$. However, this bound will always be inferior to Inequality (5.38), since $x^{SO} \geq 1$ and $\frac{e^x}{e^x + (n-1)v}$ is increasing in x.

3. A lower bound in terms of the user optimum is given by

$$LOC \geq \frac{e^{x^{UO}}}{e^{x^{UO}} + (n-1)v}.$$

However, since $x^{SO} \geq x^{UO}$ and and $\frac{e^x}{e^x + (n-1)v}$ is increasing in x, the bound in Inequality (5.38) will always be tighter than this lower bound.

4. Alternatively, from Equation (5.32), the lower bound for x^{SO} in Inequality (5.34) and the upper bound for x^{UO} in Inequality (5.33), we also have the following lower bound:

$$LOC \geq \frac{e^{1-n+nk}}{e^{\frac{n}{n-1}} + (n-1)v}.$$

Moreover, since $\frac{e^x}{e^x + (n-1)v}$ is increasing in x, and $x^{SO} \geq 1 - n + nk$ by Inequality (5.34),

$$LOC \geq \frac{e^{1-n+nk}}{e^{1-n+nk} + (n-1)v}.$$

However, the bound in Inequality (5.36) will always be better than these bounds, since $k \leq \frac{n}{n-1}$ and $k \leq \ 1 - n + nk$.

5.5 Simulation Results

In this section, we present simulation results to study the behaviour of the optimal profits, $\mathbf{x^{SO}}$ and $\mathbf{x^{SO}}$, the tightness of bounds on these profits and the tightness of bounds on the loss of coordination. This section concludes with a discussion on the behaviour of the optimal profits and loss of coordination with varying values of n and v.

5.5.1 Behaviour and Tightness of Bounds for Optimal Profits

The lower bounds for the *LOC* presented in Theorem 5.4.4 depends strongly on the lower bounds we derive for the optimal solutions, $\mathbf{x^{UO}}$ $\mathbf{x^{SO}}$, in Lemma 5.4.3. As such, we will first examine the behaviour of the optimal solutions and evaluate the tightness of the bounds presented in Lemma 5.4.3.

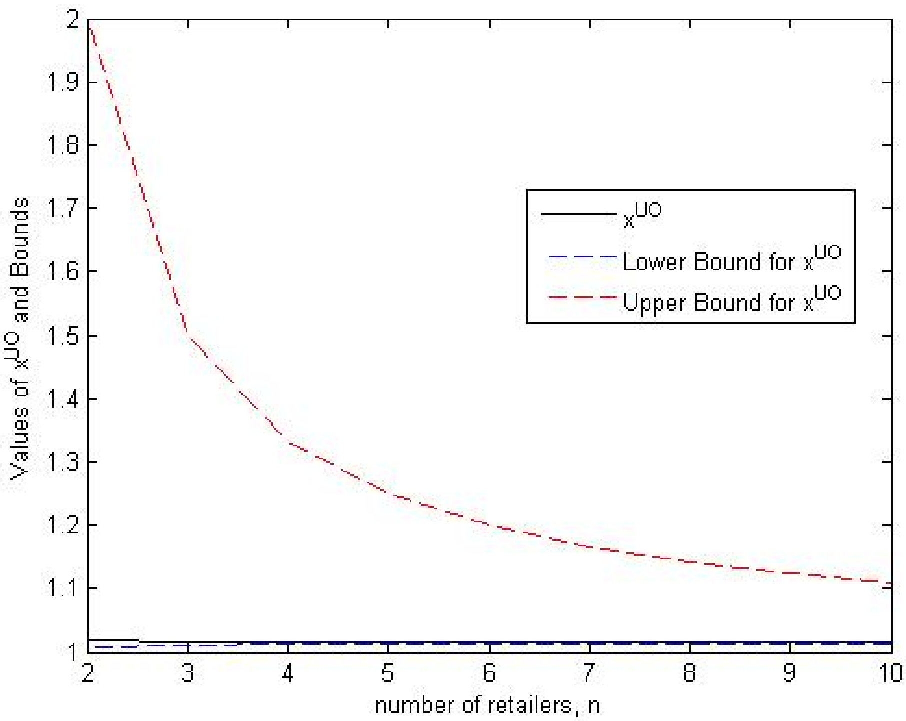

Figure 5-4: Bounds for $\mathbf{x^{UO}}$ with varying number of retailers, when $v = 0.05$.

Discussion From the simulations, we observe that

1. From Figures 5-4, 5-5 and 5-6, we see that $\mathbf{x^{UO}}$ increases as v increases, and decreases as n increases. We also see that the lower bound for $\mathbf{x^{UO}}$ is very tight, especially when v is small.

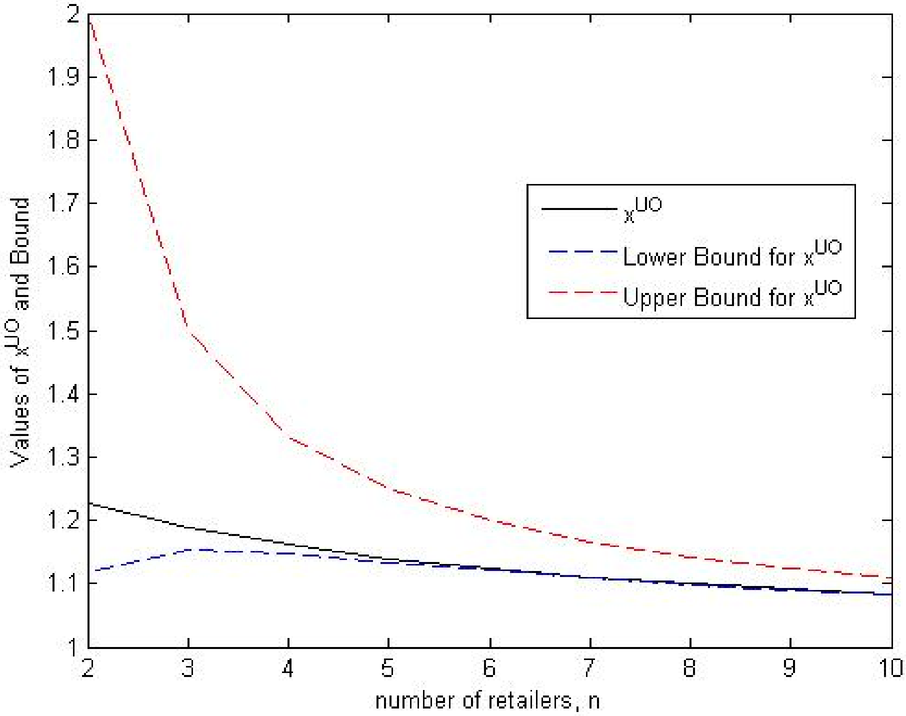

Figure 5-5: Bounds for $\mathbf{x^{UO}}$ with varying number of retailers, when $v = 1$.

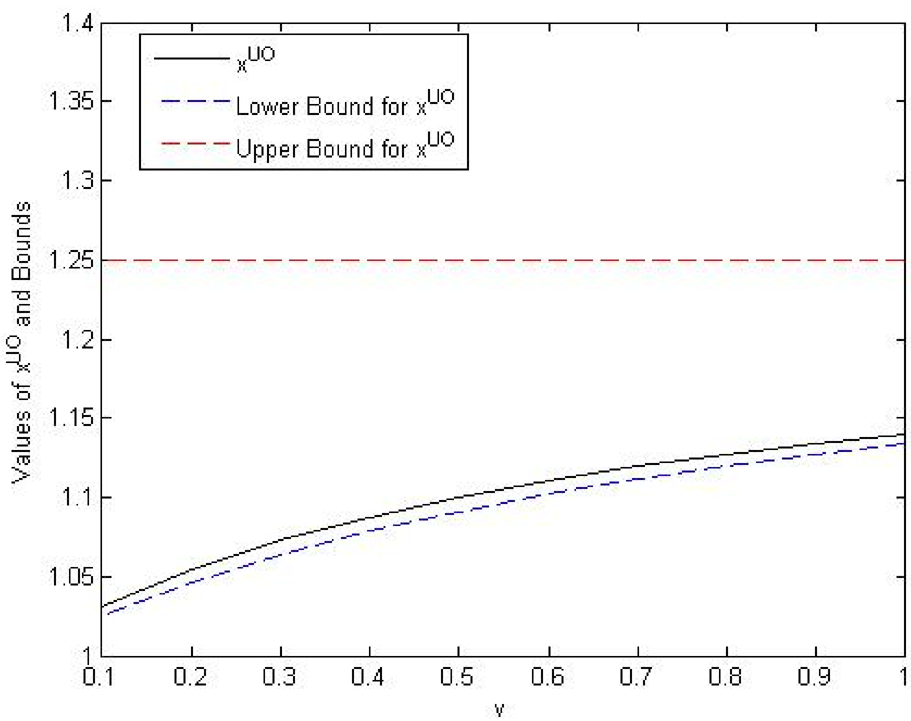

Figure 5-6: Bounds for $\mathbf{x^{UO}}$ with varying v, when $n = 5$.

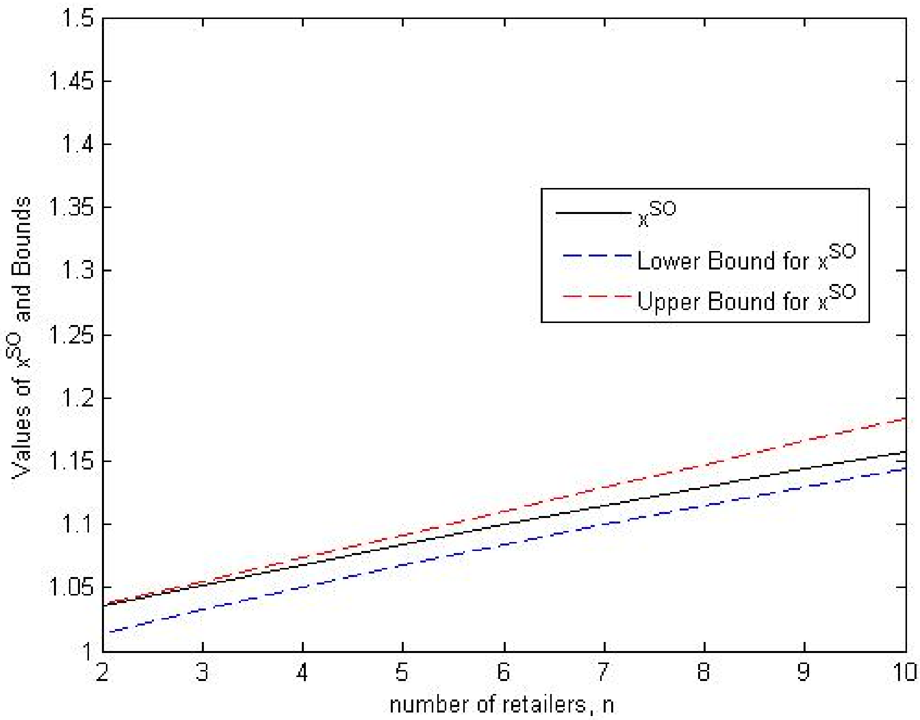

Figure 5-7: Bounds for $\mathbf{x^{SO}}$ with varying number of retailers, when $v = 0.05$.

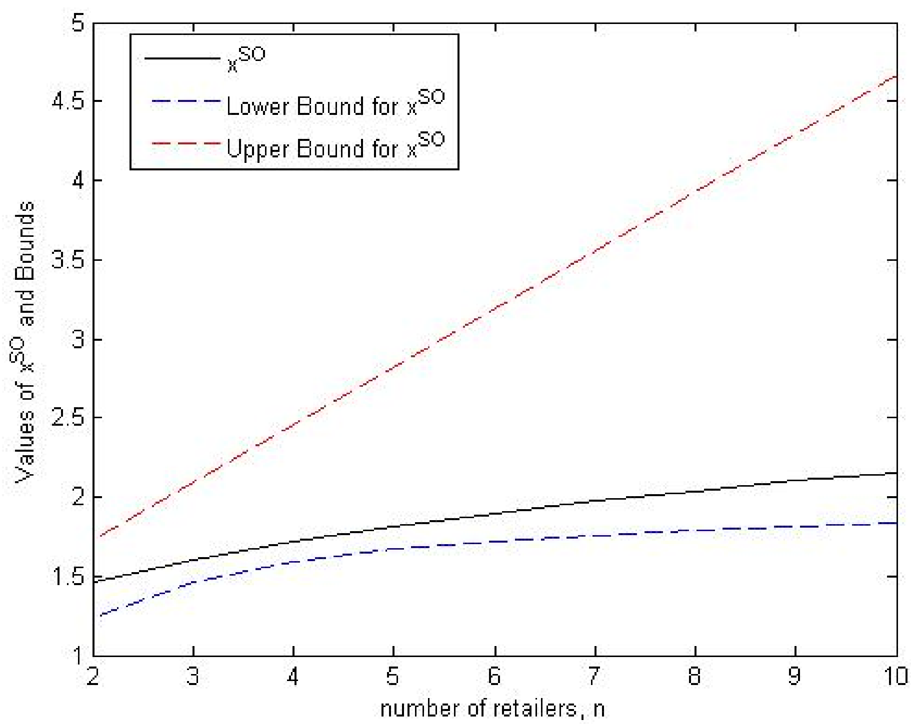

Figure 5-8: Bounds for $\mathbf{x^{SO}}$ with varying number of retailers, when $v = 1$.

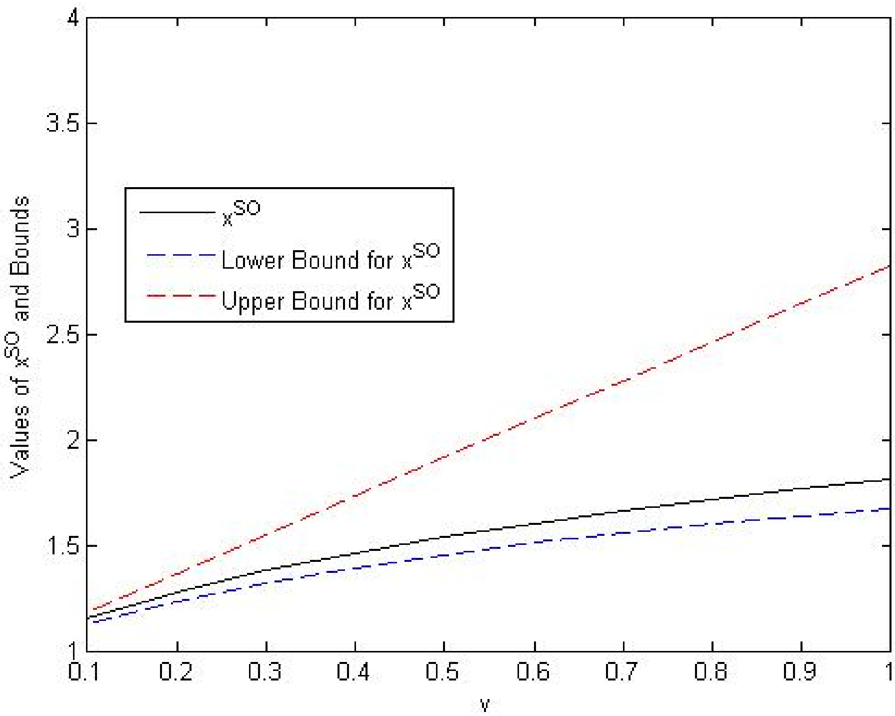

Figure 5-9: Bounds for $\mathbf{x^{SO}}$ with varying v, when $n = 5$.

2. From Figures 5-7, 5-8 and 5-9, we see that $\mathbf{x^{SO}}$ increases as v increases or as n increases. We also see that the lower bound for $\mathbf{x^{SO}}$ is very tight.

3. The behaviour of $\mathbf{x^{UO}}$ and $\mathbf{x^{SO}}$ is consistent with what we would expect from Equations (5.29) and (5.30).

We have seen that the lower bounds derived for $\mathbf{x^{UO}}$ and $\mathbf{x^{SO}}$ are very tight. This justifies the use of these lower bounds in the derivation for the lower bounds for the loss of coordination in Theorem 5.4.4. We shall examine the quality of the lower bounds for the *LOC* using simulations, presented in the following subsection.

5.5.2 Tightness of Bounds for Loss of Coordination

We analyze the tightness of the lower bounds for the loss of coordination in Theorem 5.4.4 by varying the number of retailers, n, for different values of v. First, by numerical methods, we obtain optimal solutions x^{UO} and x^{SO}, and hence the loss of coordination from Equation (5.31). We then compare the actual loss of coordination with the bounds in Inequalities (5.36) and (5.37) which only depend on constants n and v.

We considered five different values of v, that is, $v = 0.05$, 0.1, 0.4, 1 and 2. For each value of v, we vary the number of retailers, n, from 2 to 10, and evaluate the tightness of the bounds and the behaviour of the loss of coordination with respect to n.

In the following simulation, we denote Inequality (5.36) by 'Lower Bound 2' and Inequality (5.37) by 'Lower Bound 3', and plot it together with the the maximum of these two bounds shown in Theorem 5.4.4, as well as the actual loss of coordination.

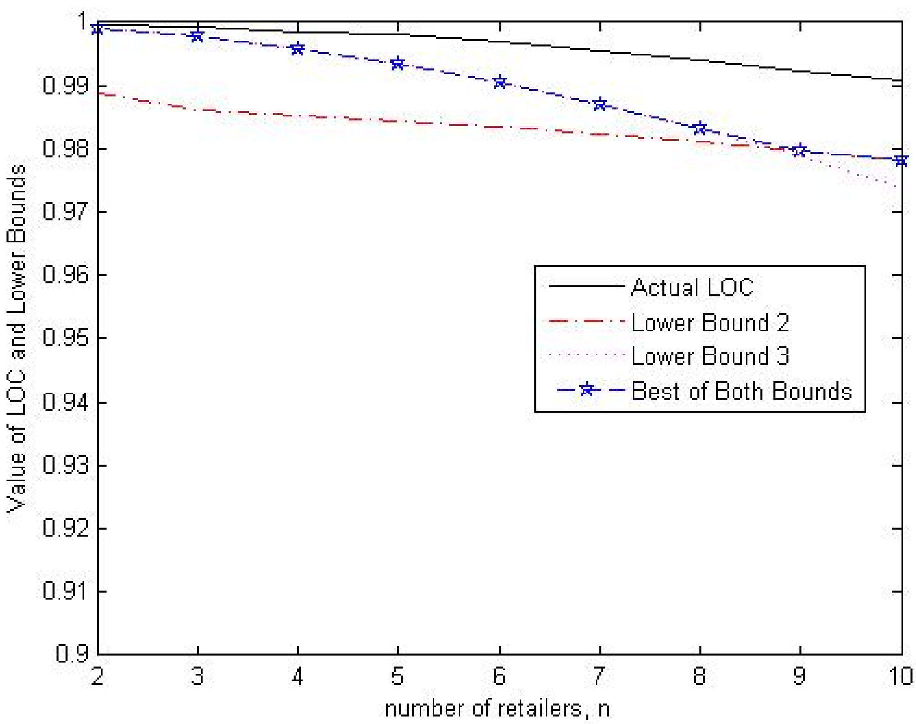

Figure 5-10: Lower bounds with varying number of retailers, when $v = 0.05$.

Discussion From the simulations, we observe that

1. A market under competition is more efficient (LOC close to 1) when there is a small number of retailers, n, and is less efficient (LOC close to 0) when the

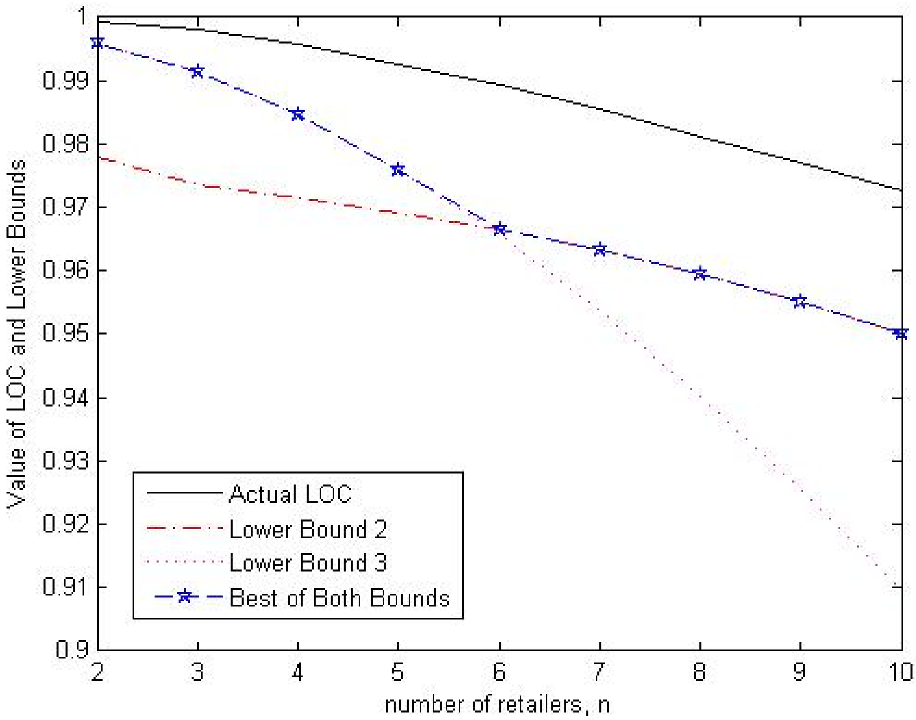

Figure 5-11: Lower bounds with varying number of retailers, when $v = 0.1$.

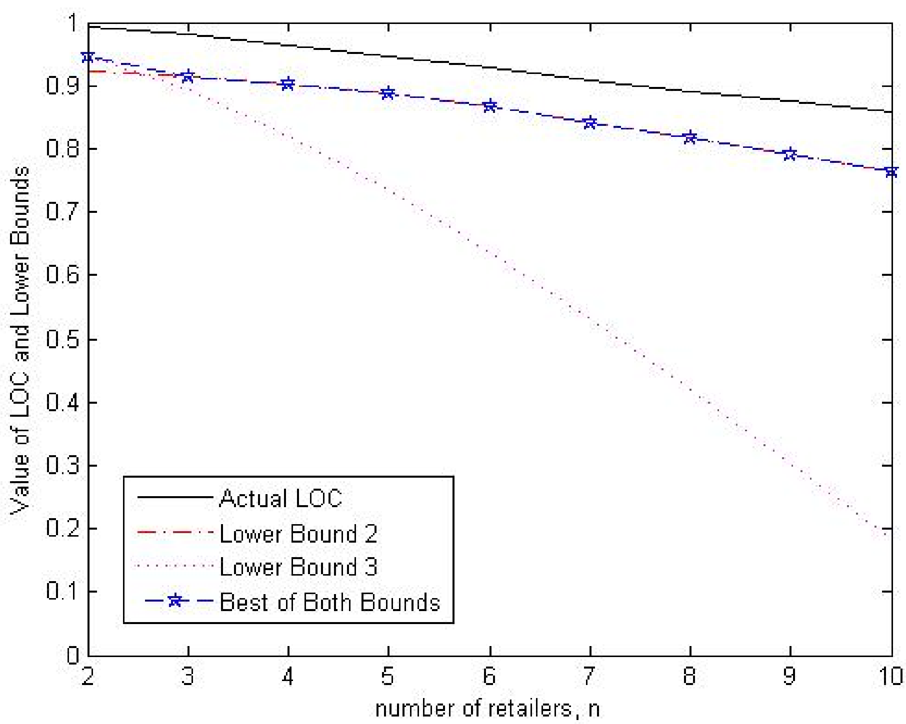

Figure 5-12: Lower bounds with varying number of retailers, when $v = 0.4$.

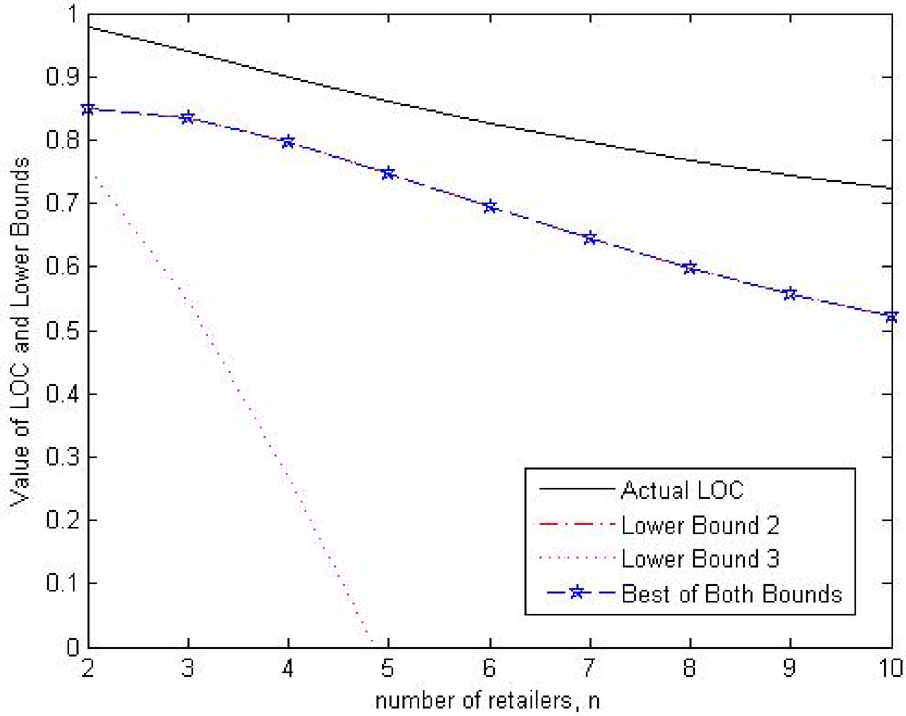

Figure 5-13: Lower bounds with varying number of retailers, when $v = 1$.

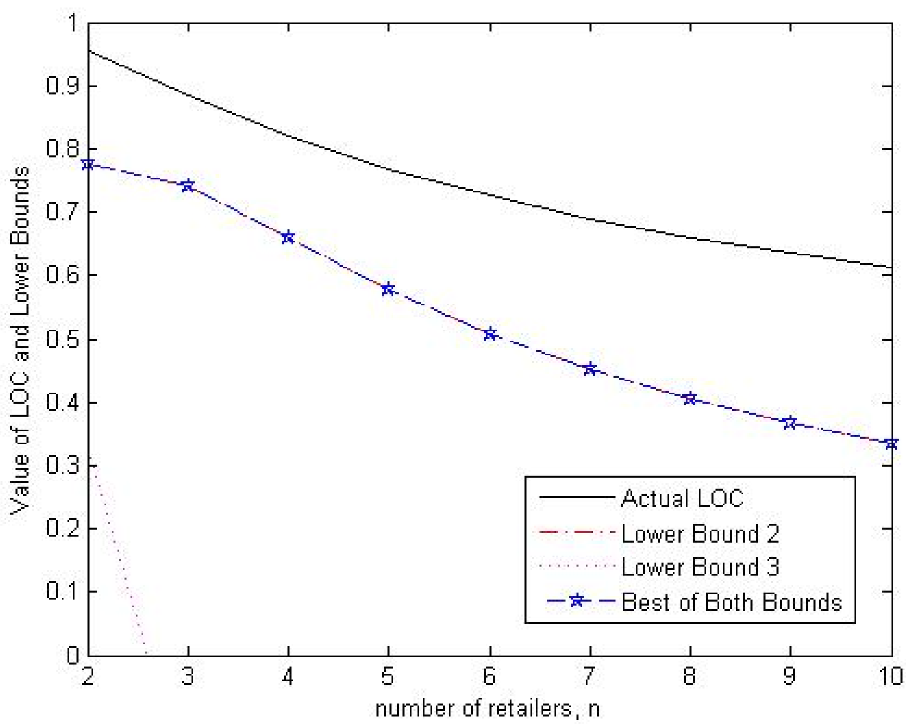

Figure 5-14: Lower bounds with varying number of retailers, when $v = 2$.

number of retailers increases. In a paper by Sun (2004), similar results for a model under a uniform affine demand in a Bertrand competition were obtained, in which the price of anarchy (which is analogous to the loss of coordination) decreases with increasing number of retailers. In our MNL model, increasing the number of retailers has a positive effect on $\mathbf{x}^{\mathbf{SO}}$ and negative effect on $\mathbf{x}^{\mathbf{UO}}$, as seen from Equations (5.29) and (5.30) as well as from simulation. Increasing the number of retailers has an effect of increasing the total market captured by these retailers, and therefore decreasing the proportion of the outside alternate, denoted by A, as seen in Equation (5.1). This increases the total profits under system optimization. However, as the number of retailers increases, competition intensifies and retailers have to keep prices low to remain competitive. Therefore, efficiency under competition decreases with increasing n.

2. A market under competition is more efficient (LOC close to 1) when v is small, and is less efficient (LOC close to 0) when v is large. Lower marginal costs or a higher weight on the determinstic component (which results from larger deterministic component, d, or lower variance, α) of the consumer utility function give rise to a larger v. Due to higher predictability of consumer behaviour, retailers are able to set a higher price. Together with the possibility of lower costs when v is large, retailers' profit under both user optimization and system optimization increases. This can be seen from Equations (5.29) and (5.30) as well as from simulations. However, such a scenario benefits the retailers under coordination more than when they are under competition, as the profit under system optimization increases faster than the profit under user optimization. This causes efficiency under competition to decrease when v increases.

3. As an example of the above remarks, we focus on the scenario when $v = 0.05$ and $v = 1$. From Figure 5-10 when $v = 0.05$, the efficiency is always above 0.99 for less than ten retailers. When $v = 1$, the efficiency drops to 0.85 for $n = 10$, as seen from Figure 5-13.

4. When v is small, say $v \leq 0.4$, Inequality (5.37) gives a better bound than

Inequality (5.36) for small number of retailers, say $n \leq 4$. As v gets larger beyond $v \geq 0.4$, Inequality (5.36) consistently gives a better bound.

5. The lower bound given in Theorem (5.4.4) is 'fairly' tight for small n and v. For example, when $v = 0.05$ in Figure 5-10, the lower bound is within 0.01 away from the actual loss of coordination, which occurs when $LOC \approx 0.99$ when $n = 10$. The lower bound is thus within 1% deviation from the actual loss of coordination. When $v = 1$ and $n = 10$ in Figure 5-13, Inequality (5.36) lower bound the LOC by approximately 0.52 when in fact, $LOC \approx 0.72$.

Chapter 6

Conclusion

In this thesis, we studied the loss of profit due to lack of coordination in a supply chain with one supplier and multiple retailers in three configurations: i) quantity competition with substitute products, ii) price competition with substitute products, and iii) quantity competition with complement products. We also looked into a single-tier oligopoly market with retailers facing the multinomial logit demand.

In the three supply chain models, we presented lower bounds for the loss of coordination in terms of two key drivers we identified - the number of retailers and price (or quantity) sensitivity for Bertrand competition (or Cournot competition, respectively). In addition, in the supply chain where retailers are in a Cournot competition offering substitute products, simulation indicates that the average loss of total profit in the chain due to competition of the supply chain is no more than 30%, implying that the uncoordinated supply chain is in fact 'fairly' efficient. Moreover, theoretical and simulation results both indicate that under uniform demand, the supply chain can be 'almost' coordinated when there is a 'reasonable' number of Cournot retailers (six or more) who have strong market power (i.e., they are able to increase supply without significantly affecting market clearing prices). In the supply chain where retailers are in a Bertrand competition offering substitute products, we observe through simulations that the maximum loss of profit due to competition of the supply chain is no more than 25%, and the average loss of profit is less than 15%. Moreover, theoretical and simulation results both indicate that under uniform demand, almost no loss in profit can be attained when demand is inelastic (i.e., retailers can increase prices without significantly decreasing demand). In the supply chain where retailers are in a Cournot competition offering complement products, we prove that the worst case scenario occurs under uniform demand, where the loss of profit due to lack of coordination is less than 25%, and remains less than 50% for a large range of values for the quantity sensitivity (i.e., for $0 \leq r \leq 0.6$). Numerical simulations indicate that, on the average, the loss of profit in the supply chain is in fact much better than the worst case scenario under uniform demand, with average loss of profit consistently below 48%.

In the last model we consider a Bertrand oligopoly of retailers facing the multinomial logit demand. We identified two key drivers - the number of retailers and predictability of consumer behaviour. We present one lower bound for the asymmetric setting and three lower bounds for the symmetric setting where all retailers encounter identical marginal costs, quality differences of products among competitors and variances in the probabilistic component of the consumers' utility function. We find that higher predictability of consumer behaviour increases profits both under coordination and under competition, and larger number of retailers decreases profits under competition, but increases profit under coordination. The net result is that efficiency of competition deteoriates with increasing number of retailers and predictability of consumer behaviour.

There are several extensions to the current model that could be proposed for future research. These include:

1. Incorporating multiple competing suppliers, and examine the loss of coordination under various degree of coordination which include:

    (a) Competing suppliers and competiting retailers and

    (b) Competing suppliers and coordinated retailers,

    and compare it with the models in this thesis which has considered

    (a) Coordinated suppliers and competiting retailers,

(b) Coordinated suppliers and coordinated retailers,

(c) Coordinated supply chain (system optimization).

2. Extend the results in the oligopoly of retailers facing the multinomial logit demand to a model which considers a two-tier supply chain, and develop better lower bounds in the asymmetric setting.

3. Use more sophisticated demand functions such as a non-linear demand function.

4. Incorporate multiple products in each of the models considered.

5. Incorporate risk adverse behaviour of retailers and uncertainty in demand in the supply chain configurations.

Bibliography

[1] Adida, E., V. DeMiguel. 2009. Efficiency and Coordination in a Supply Chain with Competing Manufacturers and Retailers. Working paper.

[2] Allon and Federgruen. 2006. Competition in service industries, *Operations Research* **55**(1) 37-55

[3] Allon and Federgruen. 2006. Outstanding service processes to a common service provider under price and time competition. Working paper.

[4] Anderson, S. P., A. de Palma, J. F. Thisse. 1992. *Discrete Choice Theory of Product Differentiation*, MIT Press, Cambridge, Massachusetts.

[5] Ben-Akiva, M. and S. Lerman. 1974. "Some Estimation Results of a Simultaneous Model of Auto Ownership and Mode Choice to Work," *Transportation*, **3** 375-376.

[6] Berkovec, J. 1985. Forecasting automobile demand using disaggregate choice models, *Transportation Research -B*, 19B-4, 315-329.

[7] Bernstein, F., A. Federgruen. 2003. Pricing and replenisment strategies in a distribution system with competing retailers. *Operations Research* **51** (3) 409-426

[8] Bernstein, F., A. Federgruen. 2005. Decentralized supply chains with competing retailers under demand uncertainty. *Management Science* **51** 18-29.

[9] Bertrand, J. 1883. Theorie mathematique de la richesse sociale, *Jornal des Savants*, **67** 499-508.

[10] Cachon, G.P. 2003. Supply chain coordination with contracts. S. Graves, T. de Kok, eds., *Handbook in Operations Research and Management Science: Supply Chain Management*. Elsevier, Amsterdam, 229-339.

[11] Cachon, G.P., M.A. Lariviere. 2005. Supply chain coordination with revenue-sharing contracts: strengths and limitations. *Management Science* **51** 30-44.

[12] Carr et al. 1999. The harmful effects of capacity improvements. Working paper.

[13] Cournot, A. 1838. Recherches sur les principes mathematics de la theorie de la richesse, *Hachettte*.

[14] Dafermos, S. C., F. T. Sparrow. 1969. The Traffic Assignment Problem for a General Network, *Journal of Research of the U.S. National Bureau of Standards.* **73B** 91118.

[15] Dubey, P. 1986. The Inefficiency of Nash Equilibria, *Mathematics of Operations Research.* vol 11, 1-8

[16] Farahat, A., G. Perakis. 2006. A comparison of Bertrand and Cournot profits in oligopolies with differentiated products; the general affine demand case, Working Paper.

[17] Farahat, A., G. Perakis. 2008. Price Competition Among Multiproduct Firms, *Johnson School Research Paper Series* No. 0608.

[18] Goundan, P. R. 2007. Essays on Optimization and Incentive Contracts. Ph.D. dissertation, Massachusetts Institute of Technology, Cambridge, Massachusetts, June 2007.

[19] Hansen, J. 2007. Higher prices but lower profits: Double marginalisation and the decentralized supply chain. *Summit, Canada's magazine on public sector purchasing.*

[20] Kluberg, J., G. Perakis. 2008. Generalized Quantity Competition for Multiple Products and Loss of Efficiency. *Communication, Control, and Computing, 2008 46th Annual Allerton Conference.* 930-936.

[21] Lariviere, M. A. and E. L. Porteus. 2001. Selling to the newsvendor: An analysis of price-only contracts, *Manufacturing & Service Operations Management.* 3:293-305.

[22] Luce, R. D. 1959. Individual choice behaviour, *Wiley: New York.*

[23] McFadden, D. 1974. Conditional Logit Analysis of Qualitative Choice Behavior, Zarembka, Paul, ed., *Frontiers in Econometrics, Academic Press: New York*, 105-142.

[24] McFadden, D. 2001. Economic Choices, *The American Economic Review*, **91** (3) 351 - 378

[25] Papadimitriou, C. H. 2001. Algorithms, Games, and the Internet, *Proceedings of the 33rd Annual ACM Symposium on the Theory of Computing*, STOC 2001 749 753.

[26] Perakis, G., G. Roels. 2007. The price of anarchy in supply chains: Quantifying the efficiency of price-only contracts. *Management Science* **53**(8) 12491268.

[27] Perakis, G., M. Zaretsky. 2008. Multiperiod Models with Capacities in Competitive Supply Chain. *Production and Operations Management.* **17** (4) 439454.

[28] Spengler, J. 1950. Vertical integration and antitrust policy, *Journal of Political Economy* **58** 347-352.

[29] Sun, W. 2005. Price of Anarchy in a Bertrand Oligopoly Market. Masters thesis, Massachusetts Institute of Technology, Cambridge, Massachusetts, June 2006.

[30] Talluri, K.T., G.J. van Ryzin. 2004. *The Theory and Practice of Revenue Management*. vol. 68 of International Series in Operations Research and Management Science. Springer.

[31] Train, K. E., L. McFadden, M. Ben-Akiva. 1987. The Demand for Local Telephone Service: A Fully Discrete Model of Residential Calling Patterns and Service Choices, *The RAND Journal of Economics*, **18** (1) 109-123.